CLEARINGHOUSE FOR FEDERAL SCIENTIFIC AND TECHNICAL INFORMATION CFSTI
DOCUMENT MANAGEMENT BRANCH 410.11

LIMITATIONS IN REPRODUCTION QUALITY

ACCESSION #

- [x] 1. WE REGRET THAT LEGIBILITY OF THIS DOCUMENT IS IN PART UNSATISFACTORY. REPRODUCTION HAS BEEN MADE FROM BEST AVAILABLE COPY.

- [ ] 2. A PORTION OF THE ORIGINAL DOCUMENT CONTAINS FINE DETAIL WHICH MAY MAKE READING OF PHOTOCOPY DIFFICULT.

- [ ] 3. THE ORIGINAL DOCUMENT CONTAINS COLOR, BUT DISTRIBUTION COPIES ARE AVAILABLE IN BLACK-AND-WHITE REPRODUCTION ONLY.

- [ ] 4. THE INITIAL DISTRIBUTION COPIES CONTAIN COLOR WHICH WILL BE SHOWN IN BLACK-AND-WHITE WHEN IT IS NECESSARY TO REPRINT.

- [ ] 5. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED, DOCUMENT WILL BE AVAILABLE IN MICROFICHE ONLY.

- [ ] 6. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED DOCUMENT WILL NOT BE AVAILABLE.

- [ ] 7. DOCUMENT IS AVAILABLE IN MICROFICHE ONLY.

- [ ] 8. DOCUMENT AVAILABLE ON LOAN FROM CFSTI (TT DOCUMENTS ONLY).

- [ ] 9.

PROCESSOR:

TSL-107-10 ⁄64

$\mathcal{H}$64-71352

AD 605 755

FTD-MT- 63-347

COPY 2 OF 3 42-P

HARD COPY $ 2.00

MICROFICHE $ 0.50

TRANSLATION

ON THE THEORY OF THE EQUATION $y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$

By

F. I. Frankl'

FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

WRIGHT-PATTERSON AIR FORCE BASE

OHIO

FOREWORD

This document is a machine translation of Russian text which has been processed by the AN/GSQ-16(XW-2) Machine Translator, owned and operated by the United States Air Force. The machine output has been fully post-edited. Ambiguity of meaning, words missing from the machine's dictionary, and words out of the context of meaning have been corrected. The sentence word order has been rearranged for readability due to the fact that Russian sentence structure does not follow the English subject-verb-predicate sentence structure. The fact of translation does not guarantee editorial accuracy, nor does it indicate USAF approval or disapproval of the material translated.

FTD-MT- 63-347

EDITED MACHINE TRANSLATION

ON THE THEORY OF THE EQUATION $y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$

BY: F. I. Frankl'

English Pages: 41

SOURCE: AN SSSR. Izvestiya. Seriya Matematicheskaya, Vol. 10, 1946, pp 135-166

S/0038-046-010-000

THIS TRANSLATION IS A RENDITION OF THE ORIGINAL FOREIGN TEXT WITHOUT ANY ANALYTICAL OR EDITORIAL COMMENT. STATEMENTS OR THEORIES ADVOCATED OR IMPLIED ARE THOSE OF THE SOURCE AND DO NOT NECESSARILY REFLECT THE POSITION OR OPINION OF THE FOREIGN TECHNOLOGY DIVISION.	PREPARED BY: TRANSLATION DIVISION FOREIGN TECHNOLOGY DIVISION WP-AFB, OHIO.

BLANK PAGE

MT-63-347.
News of the Academy of Sciences of the USSR Mathematical Series, No. 10, (1946).
Pages 135-166.

ON THE THEORY OF THE EQUATION $y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$

F. I. Frankl'

(Presented by Academician I. M. Vinogradov).

There is given the solution of two boundary value problems for the equation $y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ in the domain of the upper half-plane, adjoining a segment of the axis of abscissas. The solution is obtained by method of a double layer; here there are removed several limitations which in other works were put on the form of the boundary of the domain.

Introduction

In the given work there are solved two boundary value problems for the equation

$$y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0, \tag{1}$$

which we shall call the equation of Darboux and Tricomi, according to the names of the authors investigating it.

Solutions are sought in the region lying completely in the half-planes $y>0$, where equation (1) has an elliptic type. It is assumed that the boundary of the domain passes partially along the x axis. As for the part of the boundary, lying inside half-plane $y>0$, it is assumed only that it is sufficiently smooth and approaches the x axis at a right angle.

There will be considered the following boundary value problems:

1. Dirichlet's problem;

2. The problem in which boundary values of the unknown functions are given on the part of the boundary, lying within the half-plane $y>0$; and on the part of the boundary passing along the x axis there are given normal derivatives.

These problems were considered already by F. Tricomi in work [1], and also in papers [2] and [3]. However, the solution is given there by fairly complicated methods either with the use of the alternating method of Schwarz, or passage to the limit, proceeding from the domains, lying together with their boundaries completely within the half-plane $y>0$. In both cases Tricomi uses two-dimensional integral equations of the Fredholm type.

The second of the considered problems was reduced by S. Gellerstedt [7] to one-dimensional integral equations of the Fredholm type of a second kind by the method of a double layer and thus is solved. Here, however, Gellerstedt did make limiting assumptions about the form of the contour: it was taken that ends of arc L(Fig. 1) coincide with the arcs of a certain algebraic curve, called by Tricomi "a normal curve".

In the given work by means of corresponding estimates we remove this restriction for both problems in question.

In the Appendix we give particular solutions of the equation (1), refuting certain erroneous assertions by Tricomi, published in his article [2].

1. <u>Fundamental solution of the equation $y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$(')</u>

As any two-dimensional linear equation of elliptic type, the equation of Darboux-Tricomi can be reduced to such a form that we shall encounter second derivatives in it only in the form of the Laplace operator, and namely, during substitution of

$$y=\frac{2}{3}y^{\frac{3}{2}} \tag{1}$$

equation

$$y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \tag{2}$$

takes the form

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+\frac{1}{3y}\frac{\partial z}{\partial y}=0. \tag{3}$$

As is known, equations, containing the Laplace operator as a fundamental part, possess so-called fundamental solutions, i.e., solutions of the form

$$\begin{aligned}Z(x,y;x',y')=L(x,y;x',y')\ln[(x'-x)^2+(y'-y)^2]+\\+M(x,y;x',y'),\end{aligned} \tag{4}$$

where L and M are functions, regular in the neighborhood of point $x = x'$, $y = y'$.

We shall show that equation (3) possesses fundamental solutions Z_1 and Z_2, regular throughout half-plane $y \geq 0$, with the exception of point $x = x'$, $y = y'$, such that for every x

$$Z_1(x,0;x',y')=0, \tag{5}$$

$$\frac{\partial}{\partial y}Z_2(x,y;x',y')\Big|_{y=0}=0. \tag{6}$$

To construct the fundamental solutions Z_1 and Z_2 we shall first determine the Riemann function of equation (2). When using characteristic coordinates this equation takes the following form:

$$\frac{\partial^2 z}{\partial\xi\,\partial\eta}+\frac{1}{6(\eta-\xi)}\left(\frac{\partial z}{\partial\xi}-\frac{\partial z}{\partial\eta}\right)=0, \tag{7}$$

where

$$\xi=x-\frac{2}{3}(-y)^{\frac{3}{2}},\quad \eta=x+\frac{2}{3}(-y)^{\frac{3}{2}}. \tag{8}$$

The Riemann function in these coordinates takes the form:

$$u(\xi,\eta;\xi',\eta')=\frac{(\eta'-\xi')^{1/3}}{(\eta-\xi')^{1/6}(\eta'-\xi)^{1/6}}\,F\left(\frac{1}{6},\frac{1}{6};1;\sigma\right), \tag{9}$$

where

$$\frac{(\xi-\xi')(\eta-\eta')}{(\xi-\eta')(\eta-\xi')}=\sigma \tag{9a}$$

Hypergeometric function $F(1/6, 1/6; 1; \sigma)$ satisfies the differential equation

$$\sigma(1-\sigma)\frac{d^2u}{d\sigma^2}+\left(1-\frac{4}{3}\sigma\right)\frac{du}{d\sigma}-\frac{u}{36}=0. \tag{10}$$

However the hypergeometric differential equation

$$z(1-z)\frac{d^2u}{dz^2}+[1-(a+b+1)z]\frac{du}{dz}-abu=0 \tag{11}$$

possesses, along with hypergeometric function

$$F(a,\ b;\ 1;\ z),$$

still second independent solution,

$$\overline{F}(a,\ b;\ 1;\ z)=F(a,\ b;\ 1;\ z)\ln z+\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}+2\frac{\partial}{\partial c}\right)F(a,\ b;\ c;\ z)\Big|_{c=1}, \tag{12}$$

in accordance with which we obtain a second solution for equation (7), analogous to a Riemann function:

$$\overline{u}(\xi,\ \eta;\ \xi',\ \eta')=\frac{(\eta'-\xi')^{1/3}}{(\eta-\xi')^{1/6}(\eta'-\xi)^{1/6}}\overline{F}\left(\frac{1}{6},\ \frac{1}{6};\ 1;\ \sigma\right). \tag{13}$$

Let us consider now solutions u and $\overline{u}$ in an elliptic half-plane. If one were to introduce designations

$$\left.\begin{aligned}\rho^2&=(x'-x)^2+(y'-y)^2,\\ \rho_1^2&=(x'-x)^2+(y'+y)^2,\end{aligned}\right\} \tag{14}$$

then parameter σ turns out to be equal to

$$\sigma=\frac{\rho^2}{\rho_1^2}; \tag{15}$$

Then instead of u we shall obtain the function

$$q(x,\ y;\ x',\ y')=\frac{(2y')^{\frac{1}{3}}}{\rho_1^{1/3}}F\left(\frac{1}{6},\ \frac{1}{6};\ 1;\ \frac{\rho^2}{\rho_1^2}\right)=\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}}F\left(\frac{\rho^2}{\rho_1^2}\right) \tag{16}$$

and likewise, instead of $\overline{u}$, the function

$$\begin{aligned}\overline{q}(x,\ y;\ x',\ y')&=\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}}\overline{F}\left(\frac{1}{6},\ \frac{1}{6};\ 1;\ \frac{\rho^2}{\rho_1^2}\right)=\\ &=\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}}\left\{F\left(\frac{\rho^2}{\rho_1^2}\right)\ln\frac{\rho^2}{\rho_1^2}+G\left(\frac{\rho^2}{\rho_1^2}\right)\right\},\end{aligned} \tag{17}$$

where G(s) is a function, which is regular when $0 \leq s \leq 1$. [4]

Obviously, <u>any linear combination</u>

$$\tilde{q}=\overline{q}+cq \tag{18}$$

is a fundamental solution of equation (2).

We shall now show that constant c can be selected in such a manner that $q(x, 0; x', y')$ or, accordingly, $\left.\frac{\partial\tilde{q}}{\partial y}\right|_{y=0}$ turns into zero.

Indeed, on the basis of equation (16)

$$q(x, 0; x', y') = \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(1) = \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{5}{6}\right)} \tag{19}$$

and on the basis of (17)

$$\bar{q}(x, 0; x', y') = \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} G(1). \tag{20}$$

We shall show that G(1) has a finite value and we shall calculate it.

Let

$$F = F\,(a,\ b;\ c;\ z).$$

then, when $z = 0$,

$$\frac{\partial F}{\partial a} = \frac{\partial F}{\partial b} = \frac{\partial F}{\partial c} = 0. \tag{21}$$

On the other hand,

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2\frac{\partial}{\partial c}\right)(a+b-c) = 0. \tag{22}$$

Applying now operator $\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2\frac{\partial}{\partial c}$ to the expression

$$F(a,\ b;\ c;\ z) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} F(a,\ b;\ a+b-c+1;\ 1-z) + \\ + \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)} F(c-a,\ c-b;\ c-a-b+1;\ 1-z)\,(1-z)^{c-a-b}, \tag{23}$$

we obtain, when $a = b = 1/6$, $c = 1$, $z = 1$

$$G(1) = -2\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{5}{6}\right)}\left(\gamma + \frac{\Gamma'\left(\frac{5}{6}\right)}{\Gamma\left(\frac{5}{6}\right)}\right), \tag{24}$$

where $$\gamma = -\Gamma'(1) = \lim_{n\to\infty}\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n\right) = 0{,}577\ldots \tag{25}$$

Euler's constant.

Thus, fundamental solution

$$q_1 = \bar{q} - \frac{G(1)}{F(1)} q = \bar{q} + 2\left(\gamma + \frac{\Gamma'\left(\frac{5}{6}\right)}{\Gamma\left(\frac{5}{6}\right)}\right) q \tag{26}$$

actually turns into zero when $y = 0$.

Let us note further that

$$\left.\begin{aligned} 1-\sigma &= \frac{16}{9}\frac{y^{\frac{3}{2}}\, y'^{\frac{3}{2}}}{\rho_1^2}, \\ (1-\sigma)^{c-a-b} &= (1-\sigma)^{\frac{2}{3}} = \left(\frac{4}{3}\right)^{\frac{4}{3}} \frac{yy'}{\rho_1^{4/3}}. \end{aligned}\right\} \tag{27}$$

Then

$$q_y(x, 0; x', y') = \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} \left(\frac{4}{3}\right)^{\frac{4}{3}} \frac{y'}{\rho_1^{4/3}} \frac{\Gamma\left(-\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{6}\right)}, \tag{28}$$

$$\bar{q}_y(x, 0; x', y') = \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} \left(\frac{4}{3}\right)^{\frac{4}{3}} \frac{y'}{\rho_1^{4/3}} k, \tag{29}$$

$$k = \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2\frac{\partial}{\partial c}\right) \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)} \Bigg|_{\substack{c=1 \\ a=b=\frac{1}{6}}} = -2\frac{\Gamma\left(-\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{6}\right)}\left[\gamma + \frac{\Gamma'\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{6}\right)}\right]. \tag{29a}$$

Consequently, fundamental solution

$$q_2 = q + 2\left[\gamma + \frac{\Gamma'\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{6}\right)}\right] q \tag{30}$$

has derivative $\frac{\partial q_2}{\partial y}$, equal to zero at the x axis.

2. Reduction of Boundary Value Problems to the Case of Zero Boundary Data on the Axis of Abscissas.

Let us consider in plane (x, y) the region D, located in the half-plane $y \geq 0$ and limited:

1) by the segment of the x axis, $0 \leq x \leq 1$,

2) arc L, located in half-plane $y > 0$, with limited curvature and without angles at the ends, approaching the x axis perpendicularly (Fig. 1).

In this region we investigate two boundary value problems:

A. On arc L

$$z = f(s). \tag{1}$$

where s is the length of the arc, measured from one of the ends in planes (x, y).

The segment of the x axis

$$z = \tau(x). \tag{2}$$

Boundary values f(s) and $\tau(x)$ are assumed to be continuous and possess bounded first derivatbives; there is assumed a finite number of points of discontinuity, but under the condition that the solution for z near these points should remain limited. The function f(s) should have also second derivatives, limited by or seeking ∞ at the ends of arc thus $y^{-\epsilon}$ $(0 < \epsilon < 1)$.

B. on arc L

$$z = f(s), \tag{3}$$

On the segment of the x axis

$$\frac{\partial z}{\partial y} = \nu(x). \tag{4}$$

Function f(s) satisfies the conditions of problem A; function $\nu(x)$ is assumed continous, differentiable and satisfying the relation

$$\nu(x) = 0(1)\, x^{-\frac{2}{3}}(1-x)^{-\frac{2}{3}}. \tag{5}$$

In the neighborhood of a point in the segment of the x axis (0 x 1) the solution should remain limited.

To execute the uniqueness theorem it is necessary in this problem to add one more requirement:

Let us consider the integral

$$\int_{c_0} z\frac{dz}{dn} y^{\frac{1}{3}}\, ds, \tag{6}$$

where c_ϵ is a part of the circumference of $x^2 + y^2 = \epsilon^2$ or, accordingly, $(1-x)^2 + y^2 = \epsilon^2$, lying in domain D, dn is the differential of the normal line to this circumference, and the length of arc s is measured in plane (x, y). Then we should have

$$\lim_{\epsilon\to 0} \int_{c_\epsilon} z \frac{dz}{dn} y^{\frac{1}{3}} ds = 0. \tag{7}$$

We shall now prove that is is possible to limit ourselves to the case, where $\tau(x) \equiv 0$ and, accordingly, $\nu(x) \equiv 0$.

Let us turn to consideration of the problems.

1. Problem A. We shall select two numbers a and b, such that domain D completely lies in the zone

$$a < x < b. \tag{8}$$

We shall arbitrarily extend function $\tau(x)$ to the whole segment $a < x < b$, but in such a manner that function $\tau(x)$ remains piecewise continuous with bounded derivatives.

We shall now expand $\tau(x)$ into a Fourier series:

$$\tau(x) = \sum_{n=1}^{\infty} a_n \sin n\pi \frac{x-a}{b-a}; \tag{9}$$

then we can obtain a solution of equation (2) Section 1, determined in the whole half-plane $y > 0$ and taking on the segment of the x axis values of namely

$$\frac{1}{\lambda(0)} \sum_{n=1}^{\infty} a_n \sin n\pi \frac{x-a}{b-a} \lambda \left[n^{\frac{2}{3}} \pi^{\frac{2}{3}} \frac{y}{(b-a)^{\frac{2}{3}}} \right], \tag{10}$$

where $\lambda(\xi)$ is the solution of equation

$$\lambda''(\xi) + \xi \lambda(\xi) = 0, \tag{11}$$

given by Tricomi [1] and determined by the integral

$$\lambda(\xi) = \int_0^{\infty} e^{-\frac{1}{2}\xi\rho - \frac{1}{3}\rho^3} \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\xi\rho\right) d\rho. \tag{12}$$

Then the solution of the initially posed problem can be presented in the form

$$z(x, y) = z'(x, y) + z''(x, y), \tag{13}$$

where z"(x, y) has the segment of the x axis [0, 1] zero boundary values at all points of continuity of function $\tau(x)$. For points of discontinuity of function $\tau(x)$ it is possible to prove that in their neighborhood z'(x, y) remains bounded, and thus function z"(x, y) will possess the same property, whence, in turn, it may be concluded, that the bounded values of the functions z"(x, y) at points of discontinuity also are equal to zero. This follows from the fact that the uniqueness theorem for Dirichlet's problem can be proven on the assumption of such discontinuities. The proof is completely analogous to the proof of theorem #1 of the work of Tricomi [1].

Thus, we shall prove that function z'(x, y) in the neighborhood of interval (0, 1) of the x axis remains bounded. Where $\tau(x)$ is a continous function, this follows from the uniform convergence of series (9) [6], if one were to consider that $\lambda(\xi)$ is a positive decreasing function [7] and apply the Hardy characteristic of uniform convergence [6]. In the presence of points of discontinuity it is sufficient to consider the case where $\tau(x) = -1$ when $-1 < x < 0$, $\tau(x) = 1$ when $0 < x < 1$ etc., and to investigate the behavior of function z'(x, y) close to the origin of coordinates. In this case series (9) takes the form:

$$\tau(x) = 4\left(\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \dots\right). \tag{9a}$$

Let us consider the sum

$$\begin{aligned} &\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \dots + \frac{\sin (2n+1)\pi x}{2n+1} = \\ &= \int_0^x (\cos \pi x + \cos 3\pi x + \dots + \cos (2n+1)\pi x)\, dx = \int_0^x \frac{\sin 2(n+1)\pi x}{\sin \pi x}\, dx = \\ &= \int_0^x \sin 2(n+1)\pi x \left(\frac{1}{\sin \pi x} - \frac{1}{\pi x}\right) dx + \int_0^x \frac{\sin 2(n+1)\pi x}{\pi x}\, dx = \\ &= \int_0^x \sin 2(n+1)\pi x \left(\frac{1}{\sin \pi x} - \frac{1}{\pi x}\right) dx + \frac{1}{\pi}\int_0^{2(n+1)\pi x} \frac{\sin y}{y}\, dy. \end{aligned} \tag{14}$$

Both integrals in the right part of equation (14) obviously are bounded independently of n.

From this and from the decrease of function $\lambda(\xi)$ it follows, according to the Abel inequality [6] that the sum

$$\frac{\sin \pi x}{1} \lambda (\pi^{\frac{2}{3}} y) + \frac{\sin 3\pi x}{3} \lambda\left(\pi^{\frac{2}{3}} 3^{\frac{2}{3}} y \right) + \ldots + \frac{\sin(2n+1)\pi x}{2n+1} \lambda [\pi^{\frac{2}{3}} (2n+1)^{\frac{2}{3}} y] \tag{15}$$

also is bounded, whence follows the proven one.

If, finally, we continue function $\tau(x)$ outside of segment (0,1) so that $\tau(x)$ at points x = 0 and x = 1 remains continuous, then derivatives of the boundary values of z(x, y) on arc L will be bounded ([1] Ch. II, Sect. 6).

Thus, it is proven that in the case of problem A it is possible to limit oneself to the case $\tau(x) = 0$.

2. <u>Problem B.</u> For investigation of this problem let us consider the following solution of equation (2) Sect. 1, used already by Tricomi ([1], Ch. V, Sect, 2):

$$w(x, y; \xi) = [(x-\xi)^2 + y^2]^{-\frac{1}{6}} - |2\xi - 1|^{-\frac{1}{3}} \left[\left(x - \frac{\xi}{2\xi - 1} \right)^2 + y^2 \right]^{-\frac{1}{6}}. \tag{16}$$

Distributing the singularities of these solutions along a segment of the axis of abscissas, $0 < x < 1$, we shall obtain a solution of the following form:

$$\varphi = \int_0^1 w(x, y; \xi) \ \sigma(\xi) \, d\xi. \tag{17}$$

The boundary value of φ when y = 0 is

$$\varphi(x, 0) = \int_0^1 \{ |x - \xi|^{-\frac{1}{3}} - (x + \xi - 2\xi x)^{-\frac{1}{3}} \} \, d\xi \tag{18}$$

Boundary value of $\frac{\partial \varphi}{\partial y}$ when $y \to 0$ is

$$\varphi_y(x,0)=\lim_{y\to 0}\Big\{-\frac{2}{9}\sigma(x)\,y^2\int_0^1\Big\{\Big[(x-\xi)^2+\frac{4}{9}y^3\Big]^{-\frac{7}{6}}- \tag{19}$$

$$-|2\xi-1|^{-\frac{1}{3}}\Big[\Big(x-\frac{\xi}{2\xi-1}\Big)^2+\frac{4}{9}y^3\Big]^{-\frac{7}{6}}\Big\}d\xi-$$

$$-\frac{2}{9}y^2\int_0^1[\sigma(\xi)-\sigma(x)]\Big\{\Big[(x-\xi)^2+\frac{4}{9}y^3\Big]^{-\frac{7}{6}}-$$

$$-|2\xi-1|^{-\frac{1}{3}}\Big[\Big(x-\frac{\xi}{2\xi-1}\Big)^2+\frac{4}{9}y^3\Big]^{-\frac{7}{6}}\Big\}d\xi\Big\}.$$

If at point $\xi=x$ satisfies the Lipschits condition.

$$|\sigma(\xi)-\sigma(x)|<A(x)\,|\xi-x|\ \text{when}\ |\xi-x|<\varepsilon(x), \tag{20}$$

then

$$y^2\Big|\int_{x-\varepsilon(x)}^{x+\varepsilon(x)}[\sigma(\xi)-\sigma(x)]\Big[(x-\xi)^2+\frac{4}{9}y^3\Big]^{-\frac{7}{6}}d\xi\Big|<$$

$$<A(x)\,y^2\left(\frac{4}{9}\right)^{\frac{1}{3}}\cdot\frac{1}{y}\int_{x-\varepsilon(x)}^{x+\varepsilon(x)}\frac{d\xi}{\Big[(x-\xi)^2+\frac{4}{9}y^3\Big]^{1/6}}. \tag{21}$$

Consequently, the second component of the right side equation (19) seeks zero when $y\to 0$.

The limit of the first component will be

$$-\frac{2}{9}\sigma(x)\lim_{y\to 0}y^2\int_0^1\Big[(x-\xi)^2+\frac{4}{9}y^3\Big]^{-\frac{7}{6}}d\xi.$$

For its determination we shall introduce a new integration variable

$$t=\frac{\xi-x}{\frac{2}{3}y^{\frac{3}{2}}} \tag{22}$$

Then

$$\varphi_y(x,0)=-\frac{1}{3}\left(\frac{3}{2}\right)^{\frac{1}{3}}\sigma(x)\int_{-\infty}^{+\infty}\frac{dt}{(1+t^2)^{7/6}}=-\left(\frac{2}{3}\right)^{\frac{2}{3}}\sigma(x)\int_0^{\infty}\frac{dt}{(1+t^2)^{7/6}}. \tag{23}$$

Introducing the new variable

$$s=\frac{1}{1+t^2}, \tag{24}$$

we shall obtain

$$\int_0^{\infty}\frac{dt}{(1+t^2)^{7/6}}=\frac{1}{2}\int_0^1\frac{ds}{s^{1/6}(1-s)^{1/2}}=\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{6}\right)} \tag{25}$$

Thus,

$$\psi_y(x, 0) = -\beta\sigma(x), \tag{26}$$

where

$$\beta = \left(\frac{2}{3}\right)^{\frac{1}{3}} \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{6}\right)}. \tag{26a}$$

Consequently, solution of equation (2) Sect 1

$$\psi(x, y) = -\frac{1}{\beta}\int_0^1 w(x, y; \xi)\,\nu(\xi)\,d\xi \tag{27}$$

gives for the segment $y=0,\ 0<x<1$

$$\psi_y(x, 0) = \nu(x). \tag{27a}$$

We shall now designate by $f_1(s)$ the boundary values of function $\psi(x, y)$ on arc L. We shall prove that near the ends of arc L function $f_1(s)$ and its derivative $f'_1(s)$ remain bounded (the length of arc is measured here in the plane (x, y)).

In fact,

$$w = \rho^{-\frac{1}{3}} - \bar{\rho}^{-\frac{1}{3}}, \tag{28}$$

where

$$\left.\begin{aligned} \rho^2 &= (x-\xi)^2 + y^2, \\ \bar{\rho}^2 &= [x(2\xi-1)-\xi]^2 + y^2, \end{aligned}\right\} \tag{28a}$$

whence

$$\begin{aligned} w = \frac{[x(2\xi-1)-\xi]^2-(x-\xi)^2}{\rho^{1/3}\bar{\rho}^{1/3}[\bar{\rho}^{5/3}+\rho^{4/3}\bar{\rho}^{1/3}+\ldots]} = \frac{4\xi(1-\xi)\,x(1-x)}{\rho^{1/3}\bar{\rho}^{1/3}[\bar{\rho}^{5/3}+\ldots]} = \\ = O(1)\,\xi(1-\xi)\,\rho^{-1/3}\,(=O(1)). \end{aligned} \tag{29}$$

Since for ends of arc L

$$\left.\begin{aligned} x &= O(1)\,y^2, \\ 1-x &= O(1)\,y^2, \end{aligned}\right\} \tag{29a}$$

then

$$\xi = (\xi - x) + x = O(1)\rho + O(1)\,y^2 = O(1)\,\rho. \tag{29b}$$

Differentiating formula (29) along curve L, we obtain

$$\frac{dw}{dy} = O(1)\,\xi(1-\xi)\,\rho^{-4/3}\;(= O(1)\,\rho^{-1/3}). \tag{30}$$

Consequently,

(31)

$$f_1 = \int w(x, y; \xi)\,\nu(\xi)\,d\xi = O(1),$$

$$\frac{df_1}{dy} = O(1)\int \xi^{\frac{1}{3}}(1-\xi)^{\frac{1}{3}}\rho^{-\frac{4}{3}}\,d\xi = O(1)\int \rho^{-1}\,d\xi = O(1)\,(|\ln y| + 1), \tag{31a}$$

$$\frac{d^2 f_1}{dy^2} = O(1)\int \xi^{\frac{1}{3}}(1-\xi)^{\frac{1}{3}}\rho^{-\frac{7}{3}}\,d\xi = O(1)\int \frac{d\xi}{\rho^2} = \frac{O(1)}{y}.$$

(31b)

Thus,

$$\frac{df_1}{dy} = O(1), \tag{32a}$$

$$\frac{d^2 f_1}{dy^2} = O(1)\,y^{-\frac{1}{3}}\ln y.$$

(32b)

We calculate, finally, values for $\psi(x, y)$ and its first derivatives near points (0, 0) and (0, 1).

Near point (0, 0) $\psi = O(1)\,x(1-x)\int\limits_0^1 \xi^{\frac{1}{3}}(1-\xi)^{\frac{1}{3}}\rho^{-\frac{7}{3}}\,d\xi =$

$$= O(1)\,x(1-x)\left[x^{\frac{1}{3}}\int\limits_0^1 \rho^{-\frac{7}{3}}\,d\xi + \int\limits_0^1 \rho^{-2}\,d\xi\right] = O(1)\,x(1-x)\,R^{-1} = O(1), \tag{33}$$

where $R = x^2 + y^2$

(33a)

and $\dfrac{\partial\psi}{\partial x} = O(1)\int\limits_0^1 \xi^{\frac{1}{3}}(1-\xi)^{\frac{1}{3}}\rho^{-\frac{7}{3}}\,d\xi = O(1)\left[\int\limits_0^1 \frac{d\xi}{\rho^2} + x^{\frac{1}{3}}\int\limits_0^1 \frac{d\xi}{\rho^{7/3}}\right] = \dfrac{O(1)}{R}.$

(34)

In exactly the same way $\dfrac{\partial\psi}{\partial y} = \dfrac{O(1)}{R}.$ (34a)

Analogous calculations take place near point (0, 1).

On the basis of formulas (33) and (34) we obtain

$$\int_{c_k} z \frac{dz}{dn} y^{\frac{1}{3}} ds = O(1) R^{\frac{1}{3}}, \tag{35}$$

so that requirement (7) is satisfied.

Thus, solution of φ satisfies all conditions of problem B. Consequently, and $z-\varphi$ satisfies the same conditions, if z satisfies them.

Thus, there is proven the possibility of limitation of the case $\nu(x) \equiv 0$.

3. Reduction of Boundary Value Problems to One-dimensional Fredholm Equations of a Second Type.

On the basis of the results of the preceding paragraph we assume in the future that in problems A and B functions $\tau(x)$ and $\nu(x)$ accordingly are equal a zero.

By analogy with solution of a Dirichlet problem for a Laplace equation we shall form on the basis of fundamental solutions q_1 and q_2 dipoles, i.e., we shall take derivatives for the normal to curve L of these functions $\left(\frac{dq_1}{dn'}, \frac{dq_2}{dn'}\right)$, changing here the coordinate by the singular point. (The normal is passed in plane (x, y)).

Inasmuch as judgments in cases A and B do not in the least differ, henceforth instead of q_1 and q_2 we shall write $\hat{q}$.

The potential of the double layer, i.e., the layer of dipoles with moment $\mu(s)$, will be

$$z = \frac{1}{2} \int_0^S \mu(s') \frac{d\hat{q}}{dn'} ds'. \tag{1}$$

If differential dn' has the direction of the internal normal, then the boundary value of z on the inside of arc L will be

$$\pi\mu(s) + \frac{1}{2} \int_0^S \mu(s') \frac{d\hat{q}}{dn'} ds'. \tag{2}$$

Consequently, function $\mu(s)$ is determined from the integral equation

$$\pi\mu(s) + \frac{1}{2} \int_0^S \mu(s') \frac{d\hat{q}}{dn'} ds' = f(s), \tag{3}$$

which is an integral equation of Fredholm type of a second type.

In Sect. 4 we shall show that nucleus $\frac{d\hat{\eta}}{dn'}$ satisfies the estimate

$$\frac{d\hat{q}}{dn'} = O(1)\ln(\rho). \tag{4}$$

Consequently, to this nucleus the theory of Fredholm is applicable. It still remains to prove that the number $\frac{1}{2\pi}$ is not the characteristic number of integral equation (3).

This proposition is equivalently one that uniform integral equation

$$\pi\bar{\mu}(s) + \frac{1}{2}\int_0^s \bar{\mu}(s')\frac{d\hat{\eta}}{dn'}ds' = 0 \tag{5}$$

has no nontrivial solution.

We shall prove by reductio ad absurdum. Let $\bar{\mu}(s)$ be such a nontrivial solution. As shall be shown in Sect. 4, such a solution would satisfy the estimates

$$\bar{\mu}(s) = O(1), \quad \bar{\mu}'(s) = \frac{O(1)}{y^{1/3}}, \quad \bar{\mu}''(s) = \frac{O(1)}{y^{4/3}}, \tag{6}$$

whence it should be that the corresponding solution of the uniform boundary value problem

$$\text{or}\quad \left.\begin{array}{llll} \bar{A}:\ \bar{z}=0 & \text{on} \quad L,\ \bar{z}=0 & \text{on segment} & y=0,\ 0<x<1 \\ \bar{B}:\ \bar{z}=0 & \text{on} \quad L,\ \frac{\partial z}{\partial y}=0 & \text{on segment} & y=0,\ 0<x<1 \end{array}\right\} \tag{7}$$

satisfies the valuations

$$\bar{z} = O(1)\left[|\ln R|^2 + 1\right], \quad \frac{\partial\bar{z}}{\partial y} = O(1)\frac{|\ln R| + 1}{R}, \quad \frac{\partial z}{\partial y} = \frac{O(1)}{R^{1/3}y^{1/3}}\left[|\ln R| + 1\right], \tag{8}$$

where R is the distance of the given point from one of the ends of arc L.

But in this case we would have

$$\lim_{\varepsilon\to 0}\int_{c_\varepsilon} \bar{z}\frac{dz}{dn}y^{1/3}ds = 0, \tag{9}$$

and in this circumstance the uniqueness of the solutions of problems A and B is proven ([1], Ch. II, Sect. 7). Consequently, we would have

$$\bar{z} \equiv 0 \tag{10}$$

and then on curve L

$$\frac{d\bar{z}}{dn} \equiv 0. \tag{11}$$

Let us consider now potential $\bar{z}$, formed according to formula (1), outside curve L. By force of the continuity of normal derivatives this potential would satisfy the boundary conditions

$$\left.\begin{aligned} \bar{A}:\ & \frac{d\bar{z}}{dn} = 0 \text{ on } L,\ \bar{z} = 0 \quad \text{on } y = 0; \\ & \bar{z} = O\left(\frac{1}{R^{1/3}}\right);\ \frac{\partial \bar{z}}{\partial x}, \frac{\partial \bar{z}}{\partial y} = O\left(\frac{1}{R^{7/6}}\right) \quad \text{when } R \to \infty; \\ \bar{B}:\ & \frac{d\bar{z}}{dn} = 0 \text{ on } L,\ \frac{d\bar{z}}{dy} = 0 \quad \text{on } y = 0; \\ & \bar{z} = O\left(\frac{1}{R^{1/3}}\right);\ \frac{\partial \bar{z}}{\partial x}, \frac{\partial \bar{z}}{\partial y} = O\left(\frac{1}{R^{7/6}}\right) \quad \text{when } R \to \infty. \end{aligned}\right\} \tag{12}$$

Furthermore, near the ends of arc L conditions (8) and, consequently, (9) would be satisfied.

But under these conditions partial integration gives

$$0 = \iint \bar{z}\left(y\frac{\partial^2 \bar{z}}{\partial x^2} + \frac{\partial^2 \bar{z}}{\partial y^2}\right) dx\,dy = \left(\frac{3}{2}\right)^{\frac{1}{3}} \int_L y^{\frac{1}{3}} \bar{z} \frac{d\bar{z}}{dn}\, ds - \tag{13}$$
$$- \iint \left[y\left(\frac{\partial \bar{z}}{\partial x}\right)^2 + \left(\frac{\partial \bar{z}}{\partial y}\right)^2 \right] dx\,dy = - \iint \left[y\left(\frac{\partial \bar{z}}{\partial x}\right)^2 + \left(\frac{\partial \bar{z}}{\partial y}\right)^2 \right] dx\,dy,$$

where the double integrals extend along an infinite domain outside arc $L\,(y > 0)$. It follows from this that outside arc L we have $\bar{z} \equiv 0$.

Discontinuity of function $\bar{z}$ along arc L is equal, however, to $2\pi\bar{\mu}(s)$. Thus

$$\bar{\mu}(s) \equiv 0, \tag{14}$$

q. e. d.

Thus, the inhomogeneous integral equation (3) has a unique solution.

As will be proven in Sect 4, it satisfies estimates (6), and the corresponding potential (1) satisfies estimates (8), so that there is satisfied also condition (9), if only f(s) satisfies estimates (6).

Thus it is proven that boundary value problems A and B are really solved with the help of integral equation (3).

There remains still to prove the estimates used.

4. Proof of Estimates

For proof of the estimates mentioned in Sect. 3 there are required, first of all, certain estimates of the fundamental solution $\hat{q}$ and its derivatives for n', s', n and s where both points, determining these fundamental functions lie on contour L.

Let us remember that fundamental solution $\hat{q}$ can be written in the form

$$\hat{q} = \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} [F(\sigma) \ln \sigma + H(\sigma)], \tag{1}$$

where $\sigma = \frac{\rho^2}{\rho_1^2}$, $F(\sigma) = F\left(\frac{1}{6}, \frac{1}{6}; 1; \sigma\right)$, $H = kF + G$,

$$G = DF(a, b; c; \sigma)\Big|_{a=b=\frac{1}{6}; c=1}, \quad D = \frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2\frac{\partial}{\partial c}.$$

First of all we estimate the function

$$\hat{q}, \frac{\partial \hat{q}}{\partial n'}; \quad \frac{\partial \hat{q}}{\partial s}, \frac{\partial^2 \hat{q}}{\partial n' \partial s}; \quad \frac{\partial^2 \hat{q}}{\partial s^2}, \frac{\partial^3 \hat{q}}{\partial n' \partial s^2},$$

for which, in turn, it is required to estimate the following magnitudes:

$$F(\sigma), \; G(\sigma); \quad F', G'; \quad F'', G''; \quad F''', G''';$$

$$\frac{\partial \sigma}{\partial n'}, \frac{\partial \sigma}{\partial s}, \frac{\partial^2 \sigma}{\partial n' \partial s}, \frac{\partial^3 \sigma}{\partial n' \partial s^2}; \quad \frac{\partial}{\partial s}\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}}, \frac{\partial}{\partial n'}\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}}, \frac{\partial^2}{\partial n' \partial s}\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}},$$

$$\frac{\partial^3}{\partial n' \partial s^2}\left(\frac{2y'}{\rho_1}\right);$$

$$\frac{\partial \ln \sigma}{\partial n'}, \frac{\partial \ln \sigma}{\partial s}, \frac{\partial^2 \ln \sigma}{\partial n' \partial s}, \frac{\partial^2 \ln \sigma}{\partial s^2}, \frac{\partial^3 \ln \sigma}{\partial n' \partial s^2}.$$

To estimate functions F, G and their derivatives let us note that

$$F(\sigma) = AF_2 + (1-\sigma)^{c-a-b} BF_3, \tag{2}$$

where $F = F(a, b; c; \sigma)$, hypergeometric functions F_2 F_3 have as their argument $1-\sigma$, and coefficients A and B depend on a, b and c. Accordingly we obtain

$$G(\sigma) = D(AF_2) + (1-\sigma)^{c-a-b} D(BF_3). \tag{2a}$$

Then for the values a = b = 1/6, c = 1

$$
\left.
\begin{aligned}
F(\sigma) &= AF_2 + (1-\sigma)^{\frac{2}{3}} BF_3,\\
G(\sigma) &= D(AF_2) + (1-\sigma)^{\frac{2}{3}} D(BF_3),\\
F'(\sigma) &= CF_4 + (1-\sigma)^{-\frac{1}{3}} EF_5,\\
G'(\sigma) &= D(CF_4) + (1-\sigma)^{-\frac{1}{3}} D(EF_5),\\
F''(\sigma) &= IF_6 + (1-\sigma)^{-\frac{4}{3}} KF_7,\\
G''(\sigma) &= D(IF_6) + (1-\sigma)^{-\frac{4}{3}} D(KF_7),\\
F'''(\sigma) &= LF_8 + (1-\sigma)^{-\frac{7}{3}} MF_9,\\
G'''(\sigma) &= D(LF_8) + (1-\sigma)^{-\frac{7}{3}} D(MF_9).
\end{aligned}
\right\} \tag{3}
$$

But inasmuch as

$$
1-\sigma = \frac{4yy'}{\rho_1^2}, \tag{4}
$$

then from equations (3) follow the estimates

$$
\left.
\begin{aligned}
F(\sigma) &= O(1) = G(\sigma),\\
F'(\sigma) &= O\left[\frac{\rho_1^{\frac{2}{3}}}{y^{1/3}y'^{1/3}}\right] = G'(\sigma),\\
F''(\sigma) &= O\left[\frac{\rho_1^{8/3}}{y^{4/3}y'^{4/3}}\right] = G''(\sigma),\\
F'''(\sigma) &= O\left[\frac{\rho^{14/3}}{y^{7/3}y'^{7/3}}\right] = G'''(\sigma).
\end{aligned}
\right\} \tag{5}
$$

How derivatives of σ are estimated, we shall show by the example

$$
\frac{\partial\sigma}{\partial n'} = -\frac{\partial}{\partial n}(1-\sigma) = -\frac{4y}{\rho_1^2}\overbrace{\frac{dy'}{dn'}}^{O(y')} + \frac{8yy'}{\rho_1^4}\left[\overbrace{(x'-x)}^{O(\rho_1^2)}\frac{dx'}{dn'} + \overbrace{(y+y')}^{O(\rho_1)}\overbrace{\frac{dy'}{dn'}}^{(\rho_1)}\right] =
$$
$$
= O\left(\frac{yy'}{\rho_1^3}\right) = O\left(\frac{y'}{\rho_1}\right).
$$

Estimating similarly the remaining derivatives of σ, we obtain

$$
\left.
\begin{aligned}
&\frac{\partial\sigma}{\partial n'} = O\left(\frac{y'}{\rho_1}\right), \quad \frac{\partial\sigma}{\partial s} = O\left(\frac{y'}{\rho_1^2}\right), \quad \frac{\partial^2\sigma}{\partial s\,\partial n'} = O\left(\frac{y'}{\rho_1^2}\right),\\
&\frac{\partial^2\sigma}{\partial s^2} = O\left(\frac{y'}{\rho_1^2}\right), \quad \frac{\partial^3\sigma}{\partial s^2\,\partial n'} = O\left(\frac{y'}{\rho_1^2}\right).
\end{aligned}
\right\} \tag{6}
$$

Further

$$
\frac{\partial}{\partial n'}\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}} = \frac{1}{3y'^{\frac{2}{3}}\rho_1^{\frac{1}{3}}}\overbrace{\frac{dy'}{dn'}}^{O(y')} - \frac{y'^{\frac{1}{3}}}{\rho_1^{\frac{7}{3}}}\left[\overbrace{(x'-x)\frac{dx'}{dn'} + (y'+y)\frac{dy'}{dn'}}^{O(\rho_1^2)}\right] = O\left[\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}}\right].
$$

Estimating similarly the remaining derivatives of $\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}}$, we obtain

$$
\left.
\begin{aligned}
&\frac{\partial}{\partial n'}\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}} = O\left[\frac{y'^{1/3}}{\rho_1^{1/3}}\right], \quad \frac{\partial}{\partial s}\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}} = O\left[\frac{y'^{\frac{1}{3}}}{\rho_1^{4/3}}\right], \quad \frac{\partial^2}{\partial s^2}\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}} = O\left[\frac{y'^{\frac{1}{3}}}{\rho_1^{7/3}}\right],\\
&\frac{\partial^2}{\partial n'\,\partial s}\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}} = O\left[\frac{y'^{\frac{1}{3}}}{\rho_1^{4/3}}\right], \quad \frac{\partial^3}{\partial n'\,\partial s^2}\left[\frac{y'}{\rho_1}\right]^{\frac{1}{3}} = O\left[\frac{y'^{1/3}}{\rho^{7/3}}\right].
\end{aligned}
\right\} \tag{7}
$$

For derivative of σ we obtain

$$
\left.
\begin{aligned}
&\frac{\partial \ln \sigma}{\partial n'} = O(1), \quad \frac{\partial^2 \ln \sigma}{\partial n' \partial s} = O(1) \quad \frac{\partial^3 \ln \sigma}{\partial n' \partial s^2} = O(1), \\
&\frac{\partial \ln \sigma}{\partial s} = \frac{2}{s-s'} - \frac{2}{s+s'} + O(1), \\
&\frac{\partial^2 \ln \sigma}{\partial s^2} = -\frac{2}{(s-s')^2} + \frac{2}{(s+s')^2} + O(1).
\end{aligned}
\right\} \tag{8}
$$

Having estimates (5), (6), (7), (8) it is possible, finally, to obtain estimates for $\hat{q}$ and its derivatives, namely:

$$
\left.
\begin{aligned}
&1. \quad \hat{q} = O(1)\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}}\left[\left|\ln\frac{\rho}{\rho_1}\right| + 1\right], \\
&2. \quad \frac{\partial \hat{q}}{\partial n'} = O(1)\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}}\left[\left|\ln\frac{\rho}{\rho_1}\right| + 1\right], \\
&3. \quad \frac{\partial \hat{q}}{\partial s} = \frac{O(1)}{\rho_1}\left(\frac{y'}{y}\right)^{\frac{1}{3}}\left[\left|\ln\frac{\rho}{\rho_1}\right| + 1\right] + 2\overbrace{\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)}^{P}\left(\frac{1}{s-s'} - \frac{1}{s+s'}\right), \\
&4. \quad \frac{\partial^2 \hat{q}}{\partial n' \partial s} = \frac{O(1)}{\rho_1}\left(\frac{y'}{y}\right)^{\frac{1}{3}}\left[\left|\ln\frac{\rho}{\rho_1}\right| + 1\right] + \\
&\qquad + 2\overbrace{\frac{\partial}{\partial n'}\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)}^{Q}\left(\frac{1}{s-s'} - \frac{1}{s+s'}\right), \\
&5. \quad \frac{\partial^2 \hat{q}}{\partial s^2} = \frac{O(1)}{\rho_1 y}\left(\frac{y'}{y}\right)^{\frac{1}{3}}\left[\left|\ln\frac{\rho}{\rho_1}\right| + 1\right] + \\
&\qquad + 4\overbrace{\left[\frac{\partial}{\partial s}\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma) + \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F'(\sigma)\frac{\partial \sigma}{\partial s}\right]}^{R}\left(\frac{1}{s-s'} - \frac{1}{s+s'}\right) - \\
&\qquad - 2\overbrace{\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)}^{S}\left[\frac{1}{(s'-s)^2} - \frac{1}{(s'+s)^2}\right], \\
&6. \quad \frac{\partial^3 \hat{q}}{\partial n' \partial s^2} = \frac{O(1)}{\rho_1 y}\left(\frac{y'}{y}\right)^{\frac{1}{3}}\left[\left|\ln\frac{\rho}{\rho_1}\right| + 1\right] + \\
&+ 4\overbrace{\left\{\frac{\partial^2}{\partial s \partial n'}\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma) + \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F'(\sigma)\frac{\partial^2 \sigma}{\partial n' \partial s}\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F'(\sigma)\frac{\partial \sigma}{\partial n}\right\}}^{T} \cdot \\
&\cdot \left(\frac{1}{s-s'} - \frac{1}{s+s'}\right) - \\
&- 2\overbrace{\left\{\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F'(\sigma)\frac{\partial \sigma}{\partial n'} + \frac{\partial}{\partial n'}\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right\}}\left[\frac{1}{(s-s')^2} - \frac{1}{(s+s')^2}\right].
\end{aligned}
\right\} \tag{9}
$$

In the future there will be required other estimates of the first derivatives for s' of the coefficients in $\left(\frac{1}{s-s'} - \frac{1}{s+s'}\right)$, standing in formulas (9) and also of first and second derivatives for s' of coefficients in $\left[\frac{1}{(s-s')^2} - \frac{1}{(s+s')^2}\right]$ in the same formulas. All these estimates we take under the condition that

$$\frac{1}{2}y < y' < \frac{3}{2}y. \tag{10}$$

Then, using the above-mentioned methods, we obtain

$$\left.\begin{array}{llll} \frac{\partial P}{\partial s'} = \frac{O(1)}{y}, & \frac{\partial Q}{\partial s'} = \frac{O(1)}{y}, & \frac{\partial R}{\partial s'} = \frac{O(1)}{y^2}, & \frac{\partial S}{\partial s'} = \frac{O(1)}{y}, \\ \frac{\partial^2 S}{\partial s'^2} = \frac{O(1)}{y^2}, & \frac{\partial T}{\partial s'} = \frac{O(1)}{y^2}, & \frac{\partial U}{\partial s'} = \frac{O(1)}{y}, & \frac{\partial^2 U}{\partial s'^2} = \frac{O(1)}{y^2}. \end{array}\right\} \tag{11}$$

For the coefficients P, Q, R, S, T, U we obtain the following estimates;

$$\left.\begin{array}{lll} P = O(1)\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}}, & Q = O(1)\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}}, & R = \frac{O(1)}{\rho_1}\left(\frac{y'}{y}\right)^{\frac{1}{3}}, \\ S = O(1)\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}}, & T = \frac{O(1)}{\rho_1}\left(\frac{y'}{y}\right)^{\frac{1}{3}}, & U = O(1)\left(\frac{y'}{\rho_1}\right)^{\frac{1}{3}}. \end{array}\right\} \tag{12}$$

Using condition (10) we can reduce equations (12) to the form

$$P = O(1),\quad Q = O(1),\quad R = \frac{O(1)}{y},\quad S = O(1),\quad T = \frac{O(1)}{y},\quad U = O(1). \tag{12a}$$

Now on the basis of estimates (9), (11), (12) and (12a) we shall derive a series of estimates for integrals of the form

$$\int_0^{\mathfrak{S}} \mu(s') \frac{\partial \hat{q}}{\partial n'}\, ds'.$$

First of all we shall prove the following theorems:

<u>THEOREM I Let $\mu_1(s)$ be an integrated, bounded function; then function</u>

$$\mu_2(s) = \int_0^{\mathfrak{S}} \frac{\partial \hat{q}}{\partial n'} \mu_1(s')\, ds \tag{13}$$

<u>satisfies Hölder's condition</u>:

$$|\mu_2(s_2) - \mu_2(s_1)| \leqslant A|s_2 - s_1|^{\alpha} \quad (0 < \alpha < 1). \tag{14}$$

<u>Proof</u>. Let

$$|s_2 - s_1| = \delta, \quad s_1 < s_2.$$

Then, by (9.2) and (9.4)

$$|\mu_2(s_2) - \mu_2(s_1)| \leqslant O(1)\int_{s_1}^{s_2} \frac{ds}{y^{1/3}} \int_0^{\mathfrak{S}} \frac{|\ln|s'-s||}{s+s'}\, ds' +$$

$$+ O(1)\int_{s_1}^{s_2}\left(\int_0^{s_1-\frac{\delta}{2}} + \int_{s_2+\frac{\delta}{2}}^{\mathfrak{S}}\right)\frac{ds'}{|s'-s|}\, ds + O(1)\int_{s_1-\frac{\delta}{2}}^{s_2+\frac{\delta}{2}} |\ln|s'-s_1||\, ds' +$$

$$+O(1)\int_{s_1-\frac{\delta}{2}}^{s_1+\frac{\delta}{2}} |\ln|s'-s_1||\,ds' < O(1)\int_{s_1}^{s_2}\frac{ds}{y^{1/3}}\Big[s^{-\varepsilon'}\int_0^{\frac{s}{2}}\frac{ds'}{s'+s}+$$

$$+\frac{1}{s}\int_{\frac{s}{2}}^{\frac{3s}{2}}|s'-s|^{-\varepsilon'}\,ds'+s^{-\varepsilon'}\int_{\frac{3s}{2}}^{G}\frac{ds'}{s'+s}\Big]+O(1)\int_{s_1}^{s_2}\Big[\Big|\ln\Big(s_2+\frac{\delta}{2}-s\Big)\Big|+$$

$$+\Big|\ln\Big(s-s_1+\frac{\delta}{2}\Big)\Big|\Big]\,ds+O(1)\,(s_2-s_1)^{\varepsilon'} < O(1)\int_{s_1}^{s_2} s^{-\varepsilon'-\frac{1}{3}}\,ds+$$

$$+O(1)\,(s_2-s_1)^{\varepsilon'} < O(1)\,(s_2^{\varepsilon}-s_1^{\varepsilon})+O(1)\,(s_2-s_1)^{\varepsilon} < O(1)\,(s_2-s_1)^{\varepsilon}, \tag{15}$$

which was required.

THEOREM II. Let $\mu_2(s)$ satisfy Hölder's condition. Then function

$$\mu_3(s)=\int_0^G \frac{\partial\hat{q}}{\partial n'}\,\mu_2(s')\,ds' \tag{16}$$

is differentiable, where in the derivative satisfies the estimate

$$\mu_3'(s)=\frac{O(1)}{y^{1/3}}. \tag{17}$$

We shall prove first that the derivative exists and is equal to

$$\mu_3'(s)=\int_0^G \frac{\partial^2\hat{q}}{\partial n'\,ds}\,\mu_2(s')\,ds'. \tag{18}$$

(The integral in formula (18) is understood as the principal value in the sense of Cauchy).

For this purpose let us consider the integral

$$\mu_3(s,h)=\Big(\int_0^{s-h}+\int_{s+h}^{G}\Big)\frac{\partial\hat{q}}{\partial n'}\,\mu_2(s')\,ds'. \tag{17a}$$

whose derivative is equal to

$$\Big(\int_0^{s-h}+\int_{s+h}^{G}\Big)\mu_2(s')\frac{\partial^2\hat{q}}{\partial s\,\partial n'}\,ds'+\mu_2(s-h)\,\hat{q}_{n'}(s,s-h)-$$
$$-\mu_2(s+h)\,\hat{q}_{n'}(s,s+h). \tag{18a}$$

Function $\frac{\partial^2 \hat{q}}{\partial n' \partial s}$ has the form

$$\frac{\partial^2 \hat{q}}{\partial n' \partial s} = \frac{f(s)}{s - s'} + O(1)\left(|\ln|s - s'|| + 1\right). \tag{19}$$

Therefore

$$\begin{aligned} &\mu_s(s+\delta)\,\hat{q}_{n's}(s, s+\delta) + \mu_s(s-\delta)\hat{q}_{n',s}(s, s-\delta) = \\ &= \mu_s(s+\delta)\left[\hat{q}_{n',s}(s, s+\delta) + \hat{q}_{n's}(s, s-\delta)\right] + \\ &+ \hat{q}_{n's}(s, s-\delta)\left[\mu_s(s-\delta) - \mu_s(s+\delta)\right] = \\ &= O(1)\left[|\ln\delta| + 1\right] + O(1)\,\delta^{-1}\delta^{\alpha}, \end{aligned} \tag{20}$$

whence when $h \to 0$ there results convergence of integral (18a) to integral (18) and besides it is uniform in the neighborhood of point $s = \bar{s}$.

We note further that

$$\hat{q}_{n'}(s, s \pm h) = k(s)\ln h + O(1)\,h\left[|\ln h| + 1\right], \tag{21}$$

whence

$$\begin{aligned} &\mu_s(s-h)\,\hat{q}_{n'}(s, s-h) - \mu_s(s+h)\,\hat{q}_{n'}(s, s+h) = \\ &= \mu_s(s-h)\left[q_{n'}(s, s-h) - \hat{q}_{n'}(s, s+h)\right] + \hat{q}_{n'}\left[\mu_s(s-h) - \mu_s(s+h)\right] = \\ &= O(1)\,h\left[|\ln h| + 1\right] + O(1)\ln h \cdot h^{\alpha}. \end{aligned} \tag{22}$$

Consequently, the sum of the second and third terms in expression (18) seeks zero when $h \to 0$ and besides it is even uniform in neighborhood of point $s = \bar{s}$. Thus, formula (18) is proven.

We shall now prove estimate (17). According to equations (9.4), (11) and (12)

$$\begin{aligned} \mu_s'(s) = {} & \frac{O(1)}{y^{1/\alpha}} \int_0^{\sigma} \rho_1^{-\frac{1}{2}} \left[\left|\ln\frac{\rho}{\rho_1}\right| + 1\right] + O(1)\int_0^{\sigma} \frac{ds'}{s+s'} + \\ & + \left(\int_0^{\frac{s}{2}} + \int_{\frac{3s}{2}}^{\sigma}\right) Q\,\frac{\mu_s(s')\,ds'}{s - s'} + Q(s, s)\int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{\mu_s(s')}{s - s'}\,ds' + \\ & + Q_s'(s, s^*)\int_{\frac{s}{2}}^{\frac{3s}{2}} \mu_s(s')\,ds' = \frac{O(1)}{y^{1/\alpha}} + O(1)\left[|\ln y| + 1\right] + O(1)\left[|\ln y| + 1\right] + \\ & + O(1)\,y^{\alpha} + \frac{O(1)}{y}\cdot y = \frac{O(1)}{y^{1/\alpha}}. \end{aligned} \tag{23}$$

<u>**THEOREM III. If a continuous, bounded function $\mu_3(s)$ possesses a derivative, satisfying estimate (17), then the derivative of function**</u>

$$\mu_4(s) = \int_0^G \frac{\partial \hat{q}}{\partial n'}\, \mu_3(s')\, ds' \tag{24}$$

satisfies Hölder's condition:

$$|\mu_4'(s_2) - \mu_4'(s_1)| < \frac{A}{y_1^{1+\epsilon}} (s_2 - s_1)^{\epsilon}, \quad (y_1 < y_0,\ 0 < \epsilon < 1). \tag{25}$$

For proof we shall apply partial integration.

Let

$$\frac{\partial^2 \hat{q}}{\partial n' \partial s} = V(s, s') + Q(s, s')\left(\frac{1}{s - s'} - \frac{1}{s + s'}\right).$$

Then

$$\mu_4'(s) = \left(\int_0^a + \int_b^G\right) \frac{\partial^2 \hat{q}}{\partial n' \partial s}\, \mu_3(s')\, ds' + \int_a^b \frac{Q(s, s') - Q(s, s)}{s - s'}\, \mu_3(s')\, ds' + \tag{26}$$

$$+ Q(s, s) \int_a^b \frac{\mu_3(s')}{s - s'}\, ds' + \int_a^b \frac{Q(s, s')}{s + s'}\, \mu_3(s')\, ds' + \int_a^b V(s, s')\, \mu_3(s')\, ds' =$$

$$= \left(\int_0^a + \int_b^G\right) \frac{\partial^2 \hat{q}}{\partial n' \partial s}\, \mu_3(s')\, ds' + \int_a^b \frac{Q(s, s') - Q(s, s)}{s - s'}\, \mu_3(s')\, ds' -$$

$$- Q(s, s)\,[\mu_3(b) \ln(b - s) - \mu_3(a) \ln(s - a)] + Q(s, s) \int_a^b \mu_3'(s') \ln|s' - s|\, ds' +$$

$$+ \int_a^b \frac{Q(s, s')}{s + s'}\, ds' + \int_a^b V(s, s')\, \mu_3(s')\, ds' \quad (a < s < b). \tag{27}$$

Henceforth we shall assume that

$$a = \frac{s_1}{2}, \quad b = \frac{3s_1}{2}, \quad \delta = s_2 - s_1 < \frac{s_1}{4}. \tag{28}$$

Let us note that when $\frac{s_1}{2} < s' < \frac{3s}{2}$

$$\frac{Q(s, s') - Q(s, s)}{s' - s} = \int_0^1 Q_{s'}(s;\ s + (s' - s)\, t)\, dt = \frac{O(1)}{y_1}, \tag{29}$$

$$\frac{\partial}{\partial s} \frac{Q(s, s') - Q(s, s)}{s' - s} = \int_0^1 [Q_{s', s}(s;\ s + (s' - s)\, t) + \tag{29a}$$

$$+ (1 - t)\, Q_{s', s'}(s;\ s + (s' - s)\, t)]\, dt = \frac{O(1)}{y_1^2},$$

$$\frac{Q(s, s') - Q(s, s)}{s' - s}\Bigg|_{s_1}^{s_2} = \int_{s_1}^{s_2} \frac{Q(s, s') - Q(s, s)}{s' - s}\, ds = \frac{O(1)}{y_1^2}\, |s_2 - s_1| = \tag{29b}$$

$$= \frac{O(1)}{y_1^{1+\epsilon}}\, |s_2 - s_1|^{\epsilon},$$

and when $|s' - s_1| > \frac{s_1}{2}$

$$\frac{\partial^2 \hat q}{\partial s\, \partial n'}\Big|_{s_1}^{s_2} = \int_{s_1}^{s_2} \frac{\partial^3 \hat q}{\partial s^2\, \partial n'}\, ds = \frac{O(1)}{y_1^{4/3}} \underbrace{\int_{s_1}^{s_2} (s+s')^{-1-\varepsilon} ds}_{c\frac{(s_2-s')^{\varepsilon}(s_1+s')}{(s_1-s')^{\varepsilon}(s_2-s')} = O(1)\frac{(s_2-s_1)^{\varepsilon}}{(s_1+s')^{\varepsilon}}} + \tag{30}$$

$$+\frac{O(1)}{y_1^{4/3}} \int_{s_1}^{s_2} \frac{ds}{|s-s'|} + \frac{O(1)}{y_1^{1/3}} \int_{s_1}^{s_2} \frac{ds}{(s-s')^2} = \frac{O(1)(s_2-s_1)^{\varepsilon}}{y_1^{4/3}(s_1-s')^{\varepsilon}}.$$

Further, we have

$$\left.\begin{aligned} Q(s,s)\ln\left(\frac{3s}{2}-s\right)\Big|_{s_1}^{s_2} &= O(1)\frac{s_2-s_1}{y_1^{2}}, \\ Q(s,s)\ln\left(s-\frac{s_1}{2}\right)\Big|_{s_1}^{s_2} &= O(1)\frac{s_2-s_1}{y_1^{2}}, \end{aligned}\right\} \tag{31}$$

$$\int_{\frac{s_1}{2}}^{\frac{3s_1}{2}} \mu_3'(s')\ln|s'-s|\Big|_{s_1}^{s_2} ds' = \left(\int_{\frac{s_1}{2}}^{s_1-\delta} + \int_{s_2+\delta}^{\frac{3s_1}{2}}\right) \mu_3'(s')\ln|s'-s|\Big|_{s_1}^{s_2} ds' +$$

$$+\int_{s_1-\delta}^{s_2+\delta} \mu_3'(s')\ln|s'-s|\Big|_{s_1}^{s_2} ds' = O(1)\left(\int_{\frac{s_1}{2}}^{s_1-\delta} + \int_{s_2+\delta}^{\frac{3s_1}{2}}\right) y'^{-\frac{4}{3}} \frac{|s_2-s_1|}{|s'-s_1|} ds' +$$

$$+O(1)\int_{s_1-\delta}^{s_2-\delta} y'^{-\frac{1}{3}} \ln|s'-s_1|\, ds' + O(1)\int_{\delta}^{s_2+\delta} y'^{-\frac{1}{3}} \ln|s'-s_2|\, ds' =$$

$$= \frac{O(1)}{y_1^{4/3}}|s_2-s_1|^{\varepsilon}, \tag{32}$$

$$\frac{Q(s,s')}{s+s'} = \frac{1}{s+s'}\int_{s_1}^{s_2} \frac{\partial Q}{\partial s}\, ds - Q\int_{s_1}^{s_2} \frac{ds}{(s+s')^2} = \tag{33}$$

$$= \frac{O(1)(s_2-s_1)}{y_1^{2}} = O(1)\frac{|s_2-s_1|^{\varepsilon}}{y_1^{1+\varepsilon}},$$

$$\int_{\frac{s_1}{2}}^{\frac{3s_1}{2}} V(s,s')\Big|_{s_1}^{s_2} \mu_3(s')\, ds' = \left(\int_{\frac{s_1}{2}}^{s_1-\delta} + \int_{s_1+\delta}^{\frac{3s_1}{2}}\right)\left(\int_{s_1}^{s_2} \frac{\partial V}{\partial s}\, ds\right) ds' +$$

$$+\int_{s_1-\delta}^{s_2+\delta} V(s,s')\Big|_{s_1}^{s_2} ds' =$$

$$= O(1)\left(\int_{\frac{s_1}{2}}^{s_1-\delta} + \int_{s_2+\delta}^{\frac{3s_1}{2}}\right)\left[\frac{1}{y_1^{2}}|s'-s_1|^{1-\varepsilon'} + \frac{1}{}|s'-s_1|^{-\varepsilon'}\right] ds' +$$

$$+\frac{O(1)}{y_1}|s_2-s_1|^{1-\varepsilon'} = O(1)\frac{|s_2-s_1|^{\varepsilon}}{y_1}. \tag{34}$$

The resulting equations (27-34) give, thus:

$$\mu_4'(s_2) - \mu_4'(s_1) = O(1)\frac{|s_2-s_1|^{\varepsilon}}{y_1^{4/3}}, \tag{35}$$

q. e. d.

THEOREM IV. If $\mu_4(s)$ satisfies conditions (17) and (25), then function

$$\mu_5(s) = \int_0^{\Theta} \frac{\partial \hat q}{\partial n'} \mu_4(s')\, ds' \tag{36}$$

possesses a second derivative, satisfying the estimate

$$\mu_4''(s) = \frac{O(1)}{y_1^4 s} . \qquad (37)$$

For proof let us note that $\mu_4''(s)$ can be expressed by $\mu_4(s)$ by means of formula (27). Inasmuch as $\mu_4''(s)$ satisfies Holder's condition, formula (27) allows formal differentiation; here, in fact, there is obtained a second derivative [illegible] which is proved similar to the proof of differentiability of $\mu_4(s)$. In formula (27) let us assume that

$$a = [illegible], \quad b = \frac{3s}{2} = [illegible], \qquad (38)$$

considering these limits to be constant, on changing during differentiation.

We shall consider that when $|s'-s| > \frac{s}{2}$

$$\frac{\partial^2 q}{\partial s^2 \partial n'} = \frac{O(1)}{y^{4/\epsilon}(s+s')^\epsilon} + \frac{O(1)}{y(s+s')} + \frac{O(1)}{(s-s')^2} \qquad (39)$$

when $|s'-s| < \frac{s}{2}$

$$\frac{\partial}{\partial s}\frac{Q(s,s')-Q(s,s)}{s'-s} = \frac{O(1)}{y^2} \qquad (40)$$

$$\left.\begin{aligned} \frac{\partial}{\partial s}\left[Q(s,s)\ln\left(\frac{3s}{2}-s\right)\right] &= \frac{O(1)}{y}[\,|\ln y| + 1], \\ \frac{\partial}{\partial s}\left[Q(s,s)\ln\left(s-\frac{s}{2}\right)\right] &= \frac{O(1)}{y}[\,|\ln y| + 1], \end{aligned}\right\} \qquad (41)$$

$$\int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{\mu_4'(s')\,ds'}{s'-s} = \int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{\mu_4'(s')-\mu_4'(s)}{s'-s}\,ds' = \frac{O(1)}{y^4 s}\int_{\frac{s}{2}}^{\frac{3s}{2}} |s'-s|^{\epsilon-1} = \frac{O(1)}{y^4 s^{1-\epsilon}}, \qquad (42)$$

$$\frac{\partial}{\partial r}\frac{Q(s,s')}{s+s'} = \frac{O(1)}{y^2}, \qquad (43)$$

$$\frac{\partial}{\partial s}V(s,s') = \frac{O(1)}{y^2}[\,|\ln|s'-s|| + 1] + \overbrace{\frac{T(s,s') - T(s,s)}{s-s'}}^{\frac{O(1)}{y^2}} + \overbrace{\frac{T(s,s')}{s-s'}}^{\frac{O(1)}{y}\cdot\frac{1}{s-s'}}, \qquad (44)$$

$$\int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{\mu_4(s')}{s'-s}\,ds' = \int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{\mu_4(s')-\mu_4(s)}{s'-s}\,ds' = O(1)\,y^4. \qquad (45)$$

Hence

$$\mu_1''(s) = \frac{O(1)}{s^{4/3}}, \tag{46}$$

q. e. d.

Thus, from Theorems I--IV it follows that any solution of the homogeneous equation

$$\bar{\mu}(s) + \frac{1}{2\pi}\int_0^{\mathfrak{S}} \bar{\mu}(s')\frac{\partial \hat{\eta}}{\partial n'}\,ds' = 0 \tag{47}$$

must satisfy estimates

$$\bar{\mu}'(s) = \frac{O(1)}{s^{1/3}(\mathfrak{S}-s)^{1/3}}, \quad \bar{\mu}''(s) = \frac{O(1)}{s^{4/3}(\mathfrak{S}-s)^{4/3}}. \tag{48}$$

We pass, finally, to proof of estimates for functions

$$\bar{z},\ \frac{\partial \bar{z}}{\partial x},\ \frac{\partial \bar{z}}{\partial y},$$

where

$$\bar{z} = \frac{1}{2}\int_0^{\mathfrak{S}} \hat{q}_{n'}\bar{\mu}(s')\,ds'. \tag{49}$$

Let

$$\hat{q}_{n'} = \hat{q}_{n'1} + \hat{q}_{n'2}, \tag{50}$$

where

$$\left.\begin{aligned} \hat{q}_{n'1} &= \frac{\partial}{\partial n'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]\ln\sigma + \frac{\partial}{\partial n'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} H(\sigma)\right], \\ \hat{q}_{n'2} &= \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\frac{\partial \ln\sigma}{\partial n'}. \end{aligned}\right\} \tag{51}$$

Accordingly

$$\bar{z} = \bar{z}_1 + \bar{z}_2, \tag{52}$$

where

$$\bar{z}_1 = \frac{1}{2}\int_0^{\mathfrak{S}} \hat{q}_{n'1}\bar{\mu}(s')\,ds', \quad \bar{z}_2 = \frac{1}{2}\int_0^{\mathfrak{S}} \hat{q}_{n'2}\bar{\mu}(s')\,ds'. \tag{52a}$$

Detailed calculation shows that $\hat{q}_{n'1} = O(1)\dfrac{1+|\ln\rho|}{\rho_1}$,

(53)

whence

$$\bar{z}_1 = O(1)\int_0^G \frac{1+|\ln\rho|}{\rho_1}\,ds' = O(1)(1+|\ln R|)\overbrace{\left(\int_0^{\frac{s}{2}} + \int_{\frac{3s}{2}}^{G}\right)\frac{ds'}{\rho_1}}^{O(1)(1+|\ln R|)} + \frac{O(1)}{R}\int_{\frac{s}{2}}^{\frac{3s}{2}} |\ln|s'-s||\,ds' = O(1)(1+|\ln R|^2). \tag{54}$$

If φ' is the angle at which segment (s, s') of arc L is seen and φ_1' is the angle at which the mirror image of the segment is seen then

$$\bar{z}_2 = 2\int\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\bar{\mu}(s')\,d\varphi' - 2\int\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\bar{\mu}(s')\,d\varphi_1' = O(1). \tag{55}$$

Thus,

$$\bar{z} = O(1)(1+|\ln R|^2). \tag{56}$$

For estimate $\frac{\partial\bar{z}_1}{\partial x}$, $\frac{\partial\bar{z}_1}{\partial y}$ let us note that

$$\left.\begin{aligned}\hat{q}_{n'x1} &= \frac{\partial}{\partial x}\hat{q}_{n'1} = \hat{q}_{n'x11} + \hat{q}_{n'x12},\\ \hat{q}_{n'y1} &= \frac{\partial}{\partial y}\hat{q}_{n'1} = \hat{q}_{n'y11} + \hat{q}_{n'y12}\end{aligned}\right\} \tag{57}$$

where

$$\left.\begin{aligned}\hat{q}_{n'x11} &= \frac{\partial^2}{\partial n'\partial x}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]\ln\sigma + \frac{\partial^2}{\partial n'\partial x}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} H(\sigma)\right],\\ \hat{q}_{n'x12} &= \frac{\partial}{\partial n'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]\frac{\partial\ln\sigma}{\partial x},\\ \hat{q}_{n'y11} &= \frac{\partial^2}{\partial n'\partial y}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]\ln\sigma + \frac{\partial^2}{\partial n'\partial y}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} H(\sigma)\right],\\ \hat{q}_{n'y12} &= \frac{\partial}{\partial n'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]\frac{\partial\ln\sigma}{\partial y}.\end{aligned}\right\} \tag{57a}$$

Detailed calculation shows that

$$\left.\begin{aligned}\hat{q}_{n'x11} &= \frac{O(1)}{\rho_1^2}(1+|\ln\rho|),\\ \hat{q}_{n'y11} &= \frac{O(1)}{\rho_1^{5/3}y'^{1/3}}(1+|\ln\rho|).\end{aligned}\right\} \tag{58}$$

If we were to introduce now the designations

$$\left.\begin{aligned} \bar{z}_{x11} &= \frac{1}{2}\int_0^G \hat{q}_{n'x11}\bar{\mu}(s')\,ds', \\ \bar{z}_{y11} &= \frac{1}{2}\int_0^G \hat{q}_{n'y11}\bar{\mu}(s')\,ds', \end{aligned}\right\} \tag{59}$$

then we obtain

$$\bar{z}_{x11} = O(1)(1+|\ln R|)\left(\int_0^{\frac{s}{2}} + \int_{\frac{3s}{2}}^{G}\right)\frac{ds'}{R^2} + \tag{60}$$

$$+\frac{O(1)}{R^2}\int_{\frac{s}{2}}^{\frac{3s}{2}} \ln|s'-s|\,ds' = O(1)\,\frac{1+|\ln R|}{R}$$

and analogously

$$\bar{z}_{y11} = O(1)\,\frac{1+|\ln R|}{y^{1/2}R^{1/2}}. \tag{61}$$

We shall introduce later instead of coordinates x and y coordinate s and n (Fig. 1).

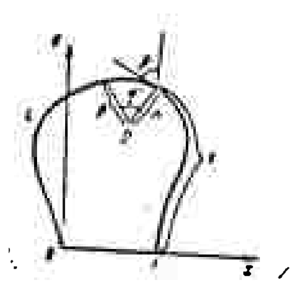

Fig. 1.

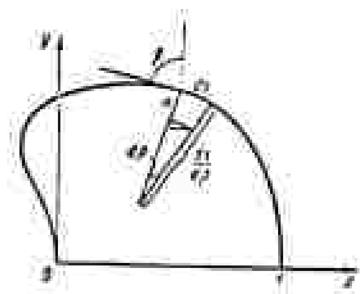

Fig. 2.

Then (Fig. 2)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial}{\partial n}\cdot\frac{\partial n}{\partial x} = \cos\beta\,\frac{\partial}{\partial n} + \frac{\sin\beta}{1-n\frac{d\beta}{ds}}\cdot\frac{\partial}{\partial s}, \tag{62}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial}{\partial n}\cdot\frac{\partial n}{\partial y} = \frac{\cos\beta}{1-n\frac{d\beta}{ds}}\cdot\frac{\partial}{\partial s} - \sin\beta\,\frac{\partial}{\partial n}, \tag{63}$$

and, consequently,

$$\left.\begin{aligned}\hat{q}_{n'x12} &= \frac{\sin\beta}{1-n\frac{d\beta}{ds}}\hat{q}_{n's12}+\cos\beta\,\hat{q}_{n'n12},\\ \hat{q}_{n'y12} &= \frac{\cos\beta}{1-n\frac{d\beta}{ds}}\hat{q}_{n's12}-\sin\beta\,\hat{q}_{n'n12},\end{aligned}\right\} \tag{64}$$

where

$$\left.\begin{aligned}\hat{q}_{n's12} &= \frac{\partial}{\partial n'}\left[\left(\frac{2\eta'}{\rho_1}\right)^{\frac{1}{3}}F(\varepsilon)\right]\frac{\partial\ln\varepsilon}{\partial s},\\ \hat{q}_{n'n12} &= \frac{\partial}{\partial n'}\left[\left(\frac{2\eta'}{\rho_1}\right)^{\frac{1}{3}}F(\varepsilon)\right]\frac{\partial\ln\varepsilon}{\partial n}.\end{aligned}\right\} \tag{65}$$

Accordingly we introduce the designations

$$\left.\begin{aligned}\tau_{s12} &= \frac{1}{2}\int_0^{\mathfrak{S}}\hat{q}_{n's12}\,\mu(s')\,ds',\\ \tau_{n12} &= \frac{1}{2}\int_0^{\mathfrak{S}}\hat{q}_{n'n12}\,\mu(s')\,ds'.\end{aligned}\right\} \tag{66}$$

We consider now that

$$\left.\begin{aligned}\frac{\partial\ln\varepsilon^2}{\partial n} &= 2\frac{\cos\gamma}{\rho}-\frac{n}{n^2+(s'-s)^2}+O(1),\\ \frac{\partial\ln\varepsilon^2}{\partial s} &= \frac{s-s'}{n^2+(s'-s)^2}+O(1),\end{aligned}\right\} \tag{67}$$

$$\frac{\partial}{\partial n'}\left[\left(\frac{2\eta'}{\rho_1}\right)^{\frac{1}{3}}F(\varepsilon)\right]=\frac{O(1)}{\rho_1}, \tag{68}$$

$$\frac{\partial^2}{\partial s'\partial n'}\left[\left(\frac{2\eta'}{\rho_1}\right)^{\frac{1}{3}}F(\varepsilon)\right]=\frac{O(1)}{\eta^{\frac{2}{3}}\rho_1^{\frac{4}{3}}}\quad\textbf{at}\quad\frac{s}{2}<s'<\frac{3s}{2}. \tag{69}$$

then

$$\begin{aligned}\tau_{n12} &= -\int_0^{\mathfrak{S}}\frac{\partial}{\partial n'}\left[\left(\frac{2\eta'}{\rho_1}\right)^{\frac{1}{3}}F(\varepsilon)\right]\frac{\partial\ln\varepsilon^2}{\partial n}\,\mu(s')\,ds'+\\ &+\left(\int_0^{\frac{s}{2}}+\int_{\frac{3s}{2}}^{\mathfrak{S}}\right)\frac{\partial}{\partial n'}\left[\left(\frac{2\eta'}{\rho_1}\right)^{\frac{1}{3}}F(\varepsilon)\right]\frac{\partial\ln\varepsilon^2}{\partial n}\,\mu(s')\,ds'+\\ &+\int_{\frac{s}{2}}^{\frac{3s}{2}}\frac{\partial}{\partial n'}\left[\left(\frac{2\eta'}{\rho_1}\right)^{\frac{1}{3}}F(\varepsilon)\right]\frac{\partial\ln\varepsilon^2}{\partial n}\,\mu(s')\,ds'=\\ &=O(1)\int_0^{\mathfrak{S}}\frac{ds'}{\rho_1}+O(1)\int_0^{\mathfrak{S}}\frac{ds'}{\rho_1}+\frac{O(1)}{R}\int_{s'=\frac{s}{2}}^{s'=\frac{3s}{2}}d\arctan\frac{s'-s}{n}+\\ &+\frac{O(1)}{R}\int_{\frac{s}{2}}^{\frac{3s}{2}}ds=\frac{O(1)}{R}.\end{aligned} \tag{70}$$

$$z_{s12} = -\int_0^{\mathfrak{S}} \frac{\partial}{\partial n'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right] \frac{\partial \ln \rho_1^2}{\partial s}\, \mu(s')\, ds' + \tag{71}$$

$$+\left(\int_0^{\frac{s}{2}} + \int_{\frac{3s}{2}}^{\mathfrak{S}}\right) \frac{\partial}{\partial n'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right] \frac{\partial \ln \rho^2}{\partial s}\, \mu(s')\, ds' -$$

$$-\int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{\partial}{\partial n'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right] \frac{(s'-s)^2}{n^2+(s'-s)^2}\, \bar{\mu}'(s^*)\, ds' -$$

$$-\int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{\partial^2}{\partial s'\,\partial n'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]_{s'=s^{**}} \frac{(s'-s)^2}{n^2+(s'-s)^2}\, \mu(s')\, ds',$$

where

$$\left.\begin{matrix} s < s^* < s' \quad \text{or} \quad s' < s^* < s; \\ s < s^{**} < s' \quad \text{or} \quad s' < s^{**} < s. \end{matrix}\right\} \tag{71a}$$

Let us note that $\frac{(s'-s)^2}{n^2+(s'-s)^2}$ is an increasing function of $|s'-s|$, so that

$$\frac{(s'-s)^2}{n^2+(s'-s)^2} = O(1)\,\frac{y^2}{R^2} \tag{72}$$

and

$$\bar{z}_{s12} = O(1)\int_0^{\mathfrak{S}} \frac{ds'}{\rho_1^2} + O(1)\int_0^{\mathfrak{S}} \frac{ds'}{\rho_1^2} + \frac{O(1)}{R}\,\frac{y^2}{R^2}\,\frac{y}{y^{1/3}} + \frac{O(1)}{R^{1/3}\, y^{1/3}}\,\frac{y^2}{R^2} = \frac{O(1)}{R}. \tag{73}$$

Since

$$\left.\begin{aligned} \bar{z}_{x12} &= \frac{\sin\beta}{1 - n\frac{d\beta}{ds}}\,\bar{z}_{s12} + \cos\beta\,\bar{z}_{n12}, \\ \bar{z}_{y12} &= \frac{\cos\beta}{1 - n\frac{d\beta}{ds}}\,\bar{z}_{s12} - \sin\beta\,\bar{z}_{n12}, \end{aligned}\right\} \tag{74}$$

that from equations (70) and (73) it follows that

$$\left.\begin{aligned} z_{x12} &= \frac{O(1)}{R}, \\ z_{y12} &= \frac{O(1)}{R}. \end{aligned}\right\} \tag{75}$$

And, finally, formulas (60), (61) and (75) will give

$$\left.\begin{aligned}\frac{\partial \bar z_1}{\partial x} &= O(1)\,\frac{1+|\ln R|}{R},\\ \frac{\partial \bar z_1}{\partial y} &= O(1)\,\frac{1+|\ln R|}{y^{1/3}R^{2/3}}.\end{aligned}\right\} \tag{76}$$

In order to estimate derivatives of $\frac{\partial \bar z_2}{\partial x}$ and $\frac{\partial \bar z_2}{\partial y}$, we shall express them by means of formulas (62) and (63) by $\frac{\partial \bar z_2}{\partial n}$ and $\frac{\partial \bar z_2}{\partial s}$.

We shall first estimate $\frac{\partial \bar z_2}{\partial n}$:

$$\frac{\partial \bar z_2}{\partial n} = \left(\int_0^{\frac{s}{2}} + \int_{\frac{3s}{2}}^{S}\right) \frac{\partial}{\partial n}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right] \frac{\partial \ln \rho^2}{\partial n'}\,\mu(s')\,ds' + \tag{77}$$

$$+2\,\frac{\partial}{\partial n}\int_{\bar\varphi}^{\bar{\bar\varphi}} \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\bar\mu(s')\,d\varphi' - \int_0^S \frac{\partial}{\partial n}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right] \frac{\partial \ln \rho_1^2}{\partial n'}\,\bar\mu(s')\,ds',$$

where $\bar\varphi$ and $\bar{\bar\varphi}$ are values of φ', corresponding to $s' = \frac{s}{2}$ and $s' = \frac{3s}{2}$.

The first and third integrals of the right side of equation (77) are estimated as $\frac{O(1)}{R}$. For the second we obtain the expression

$$\frac{\partial}{\partial n}\int_{\bar\varphi}^{\bar{\bar\varphi}} \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\bar\mu(s')\,d\varphi' = \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\bar\mu(s')\,\frac{d\varphi'}{dn}\bigg|_{s'=\frac{s}{2}}^{s'=\frac{3s}{2}} + \tag{78}$$

$$+\int_{\bar\varphi}^{\bar{\bar\varphi}} \frac{\partial}{\partial n}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]\bar\mu(s')\,d\varphi' +$$

$$+\int_{\bar\varphi}^{\bar{\bar\varphi}} \frac{\partial}{\partial s'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]\bar\mu(s')\,\frac{ds'}{dn}\bigg|_{\varphi'=\mathrm{const}} d\varphi' +$$

$$+\int_{\bar\varphi}^{\bar{\bar\varphi}} \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\bar\mu'(s')\,\frac{ds'}{dn}\bigg|_{\varphi'=\mathrm{const}} d\varphi'.$$

We have (Fig. 3)

$$\frac{d\varphi'}{dn}\bigg|_{s'=\mathrm{const}} = -\frac{\sin\varphi'}{\rho}. \tag{79}$$

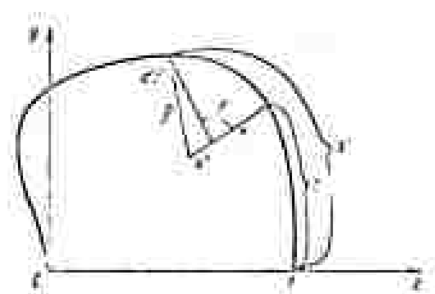

Fig. 3.

Consequently, the first term of the right side of equation (78) is estimated as $\frac{O(1)}{R}$. The same goes for the second term. Further (Fig. 4)

$$\left.\frac{ds'}{dn}\right|_{\varphi'=\text{const}} = \frac{\sin\varphi'}{\cos\varphi'}, \tag{80}$$

$$\frac{ds'}{dn}\,d\varphi' = \frac{\sin\varphi'\cos\varphi}{\cos\varphi\quad\rho}\,ds' = \left[\frac{s'-s}{n^2+(s'-s)^2} + O(1)\right] ds'. \tag{81}$$

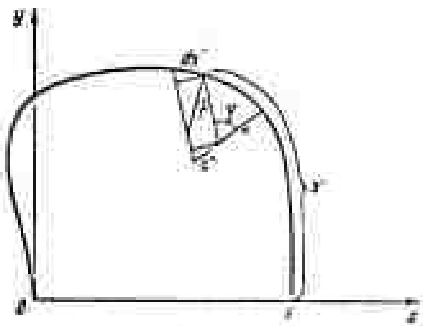

Fig. 4.

Consequently,

$$\frac{\partial}{\partial n}\int_{\varphi}^{\bar{\varphi}} \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\bar{\mu}(s')\,d\varphi' = \frac{O(1)}{R} + \frac{O(1)}{R} + \tag{82}$$

$$+\int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{\partial}{\partial s'}\left\{\frac{\partial}{\partial s'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]\bar{\mu}(s')\right\}\Big|_{s'=s} \frac{(s'-s)^2}{n^2+(s'-s)^2}\,ds' +$$

$$+O(1)\int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{ds'}{\rho_1} + \int_{\frac{s}{2}}^{\frac{3s}{2}} \frac{\partial}{\partial s'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\bar{\mu}'(s')\right]\frac{(s'-s)^2}{n^2+(s'-s)^2}\,ds' + \frac{O(1)}{y^{1/3}}\int_{\frac{s}{2}}^{\frac{3s}{2}} ds'.$$

But

$$\frac{\partial^2}{\partial s'^2}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right] = \frac{O(1)}{y^{5/3}\rho_1^{1/3}} = \frac{O(1)}{y^{5/3}R^{1/3}} \text{ where } \frac{s}{2} < s' < \frac{3s}{2}, \tag{83}$$

so that

$$\frac{\partial}{\partial n}\int_{\varphi}^{\bar{\varphi}} \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\,\bar{\mu}(s')\,d\varphi' = \frac{O(1)}{R} + \frac{O(1)}{R^{1/3}y^{5/3}}\frac{y^2}{R^2}\cdot y + \tag{84}$$

$$+\frac{O(1)}{R^{1/3}y^{2/3}}\frac{1}{y^{1/3}}\cdot\frac{y^2}{R^2}\cdot y + O(1)\left(|\ln R| + 1\right) + \frac{O(1)}{y^{1/3}}\cdot\frac{1}{y^{2/3}}\frac{1}{R^{1/3}}\frac{y^2}{R^2}\cdot y +$$

$$+\frac{O(1)}{y^{1/3}}\cdot\frac{y^2}{R^2}\cdot y + O(1)\,y^{\frac{2}{3}} = \frac{O(1)}{R}.$$

Thus, from equations (77) and (84) it follows that

$$\frac{\partial \bar{z}_1}{\partial n} = \frac{O(1)}{R}. \tag{85}$$

Let us turn to the estimate of $\frac{\partial \bar{z}_1}{\partial s}$:

$$\frac{\partial \bar{z}_1}{\partial s} = \left(\int_0^{\frac{s}{2}} + \int_{\frac{3s}{2}}^{G}\right) \frac{\partial}{\partial s}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\frac{\partial \ln \sigma}{\partial n'}\right]\bar{\mu}(s')\,ds' + \tag{86}$$

$$+\frac{\partial}{\partial s}\int_{\varphi}^{\bar{\varphi}}\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\bar{\mu}(s')\,d\varphi' = \frac{O(1)}{R} + \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\bar{\mu}(s')\frac{d\varphi'}{ds}\Big|_{s'=\frac{s}{2}}^{s'=\frac{3s}{2}} +$$

$$+\int_{\varphi}^{\bar{\varphi}}\frac{\partial}{\partial s}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\right]\bar{\mu}(s')\,d\varphi' +$$

$$+\int_{\varphi}^{\bar{\varphi}}\frac{\partial}{\partial s'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\bar{\mu}(s')\right]\frac{ds'}{ds}\Big|_{\varphi'=\text{const}}\,d\varphi'.$$

We have

$$\frac{d\varphi'}{ds}\Big|_{s'=\text{const}} = \left(1 - n\frac{d\beta}{ds}\right)\frac{\cos\varphi'}{\rho}, \tag{87}$$

$$\frac{ds'}{ds}\Big|_{\varphi'=\text{const}} = \left(1 - n\frac{d\beta}{ds}\right)\frac{\cos\varphi'}{\cos\varphi}, \tag{88}$$

$$\frac{ds'}{ds}\Big|_{\varphi'=\text{const}} d\varphi' = \left(1 - n\frac{d\beta}{ds}\right)\frac{\cos\varphi'}{\rho}\,ds' =$$

$$= \left[\frac{n}{n^2 + (s'-s)^2} + O(1)\right] ds', \tag{88a}$$

$$\bar{\varphi} - \varphi = O(1)\frac{y}{R}. \tag{89}$$

Consequently,

$$\frac{\partial \bar{z}_1}{\partial s} = \frac{O(1)}{R} + \frac{O(1)}{R} + \frac{y}{y^{1/3}R^{2/3}}\cdot\frac{y}{R} + \tag{90}$$

$$+\left(1 - n\frac{d\beta}{ds}\right)\int_{s'=\frac{s}{2}}^{s'=\frac{3s}{2}}\frac{\partial}{\partial s'}\left[\left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F(\sigma)\bar{\mu}(s')\right]\left[O(1)\,ds' + d\arctan\frac{s'-s}{n}\right] =$$

$$= \frac{O(1)}{R} + \frac{O(1)}{y^{1/3}R^{2/3}}\cdot\frac{y}{R} = \frac{O(1)}{R}.$$

From equations (62, (63), (85) and (90) we obtain now the sought for estimates of the derivatives:

$$\frac{\partial \bar z}{\partial s} = \frac{O(1)}{R}, \quad \frac{\partial \bar z}{\partial t} = \frac{O(1)}{R}. \tag{91}$$

Finally, equations (76) and (91) give

$$\left.\begin{aligned} &[illegible] = O(1)\,\frac{\ln R}{R}, \\ &[illegible] = O(1)\,\frac{\ln R}{R\,[illegible]}. \end{aligned}\right\} \tag{92}$$

Thus, are derived all estimates by which in the preceding paragraph it was proven that $\mu(s) = 0$.

Further, on the basis of Theorems I—IV, it is easy to show that estimates, derived initially for $\bar\mu$ and $\bar z$, take place also for μ and

$$z = \frac{1}{2}\int_0^{\sigma} \dot q_{n'}\,\mu(s')\,ds',$$

from which it follows that $z(x, y)$ belongs to the class of those solutions, for which the uniqueness theorem is proven.

Appendix

On Certain Particular Solutions of the Equation $y\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

We shall give examples of solutions of the equation $y\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ which refute three erroneous theorems contained in the article by Tricomi [2]

The first of these theorems ([2], Sect 2) states:

The solution of equation $y\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, cannot remain bounded in the neighborhood of an isolated effectively singular point, belonging to the x axis.

Tricomi calls a point "effectively singular," if it cannot be turned into a regular one, changing the value of the function only of this point. Here Tricomi a priori limits himself to consideration of half-plane $y > 0$; it is in this sense that the isolation of a singular point, is understood.

The theorem is refuted by the example of the function

$$z = T^{*}\left(\frac{4}{9}\frac{y^3}{x^2}\right). \; *$$

*This function was given by Tricomi ([2], Sect. 3) in connection with another problem.

where

$$T^*(t) = \int_0^t \frac{ds}{s^{2/3}(1+s)^{5/6}} \quad \left(= 2\int_0^{\theta} \frac{d\theta}{\sin^{1/3}\theta},\ \theta = \operatorname{arctan}\sqrt{t}\right),$$

Really, for this function

$z = 0$ when $y = 0$, $x > 0$ and

$$z = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}\right)} \quad \text{when } y = 0,\ x < 0,$$

and at all remaining points of half-plane $y > 0$ the value of z lies between the two indicated values.

Derivatives of $\frac{\partial z}{\partial y}$ are finite and are continuous along the whole x axis with the exception of the origin of coordinates. Indeed,

$$\left.\frac{\partial z}{\partial y}\right|_{y=0} = \frac{2 \cdot 3^{1/3}}{x^{2/3}},$$

Error can be corrected, if to the prerequisites of the theorem there is added the requirement that in the mentioned isolated singular point function z(x, 0) remains continuous. Then the proof of Tricomi will remain in force.

Regarding the two remaining theorems, then apparently, even with more precise definition of the prerequisites they remain erroneous.

The second theorem of ([2], Sect. 2) states:

Let there be given a solution z of equation $y\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \ (y > 0)$, regular in the neighborhood of the origin of coordinates, which differs at this point (i.e., with any approximation to this point seeks either $\kappa + \infty$, or $\kappa - \infty$). Then

$$z = z_0 + a\rho^{-\frac{1}{3}} \quad \left(\rho = \sqrt{x^2 + \frac{4}{9}y^3}\right),$$

where a is a constant and z_0 is a function, regular at the origin of coordinates, if function $\nu(x) = z\,(x, 0)$ is integratible in the neighborhood of point x = 0.

This preposition is refuted by example of the function

$$z = \rho^{-\frac{1}{6}} F\left[\frac{1}{6},\ \frac{1}{6};\ \frac{2}{3};\ \frac{1}{2}\left(1+\frac{x}{\rho}\right)\right].^{*}$$

This solution differs at the origin of coordinates, but cannot be presented in the form $z = z_0 + a\rho^{-\frac{1}{3}}$. On the x axis $(x \neq 0)$ it remains regular, since

$$z = \frac{2}{\sqrt{3}} x^{-\frac{1}{6}}, \quad \frac{\partial z}{\partial y} = -\frac{2\pi}{3^{1/2}\,\Gamma^{2}\left(\frac{1}{6}\right)} x^{-\frac{5}{6}} \qquad \text{when} \quad x > 0,$$

$$z = (-x)^{-\frac{1}{6}}, \quad \frac{\partial z}{\partial y} = 0 \qquad \text{when} \quad x < 0.$$

Function $\nu(x)$, as it is easy to see, is integrated in the neighborhood of point x = 0.

The third theorem ([2], Sect 4) states:

<u>The necessary and sufficient condition of the existence of a solution of the Cauchy problem for equations of $y\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ in the regions $y > 0,\ x^2 + \frac{4}{9}y^3 < 1$ is the analyticity of function</u>

$$f_1(x_0) = \tau(x_0) + \gamma \int_{-1}^{1} \left[\,|x - x_0|^{-\frac{1}{3}} - (1 - x x_0)^{-\frac{1}{3}}\right] \nu(x)\, dx$$

<u>on segment $y = 0,\ -1 < x < 1$, where</u>

$$\tau(x) = z(x, 0), \quad \nu(x) = z_y(x, 0), \quad \gamma = \frac{3^{\frac{2}{3}}\,\Gamma^{3}\left(\frac{1}{3}\right)}{[illegible]}.$$

In reality this condition is necessary, but not sufficient, as it is easy to establish by the example of the fundamental solution brought by us for

$$q = \left(\frac{2y'}{\rho_1}\right)^{\frac{1}{3}} F\left(\frac{1}{6},\ \frac{1}{6};\ 1;\ \frac{\rho^2}{\rho_1^2}\right)$$

when y' = 1/2, x' = 0.

I consider it necessary to make certain corrections and supplements to my preceding works.

<u>In the work "On the Cauchy problem"</u>, No. 8 (1944), 195--224.

Formula (10) on page 197 should be replaced by the following

---

*The example is taken from a series of solutions of equation [illegible] of main memoirs of Tricomi ([1], Ch. III, Sect 4).

$$y' = \left(\frac{3}{2}\int_0^y \sqrt{f(y)}\,dy\right)^{2/3}. \tag{10}$$

In (120) on page 218, and also in (120a—120f) on page 219 one should replace factor $(\mu-\lambda)^{-1/3}$ with $(\mu-\lambda)^{-2/3}$, and decrease the exponents of the degree of the difference $\mu_0-\lambda_0$ by 1/3.

<u>In the work "Toward a theory of Laval nozzles"</u> No. 9 (1945), 387—422.

In (15) and (16) on page 416 one should replace component $\frac{1}{2\sqrt{3}}$ by $\frac{2}{3\sqrt{3}}$. In accordance with this in (19) on page 416, (5), (5a), (8), (8a) on page 417 the factor $\sqrt{3}$ stands not in the numerator but in the denominator. In (3) on page 416 one should change the signs of the first terms of the right sides. Function $\lambda(\xi)$ (Sect. 6, (14)) can be expressed by the Bessel functions (see [8]):

$$\lambda(\xi) = \frac{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)}{\pi} K_{\frac{1}{3}}\left(\frac{2}{3}\xi^{\frac{3}{2}}\right)\sqrt{\xi}.$$

As S. V. Falkovich showed me (see also in book [9], page 58), the hypergeometric function F(1/3, —1/3; 1/2; t) is algebraic:

$$F\left(\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}; t\right) = \cos\frac{2}{3}\arcsin\sqrt{t} =$$
$$= \frac{1}{2}\left[\left(\sqrt{-t}+\sqrt{1-t}\right)^{\frac{2}{3}} + \left(\sqrt{-t}+\sqrt{1-t}\right)^{-\frac{2}{3}}\right];$$

in connection with this conclusions of paragraphs 3 and 4 can be considerably simplified.

Submitted
4 October 1945.

<u>Literature</u>

1. F. Tricomi. Sulle equazioni lineari alle derivate parziali di secondo ordine di tipo misto, Memorie della Reale Accademia Nazionale dei Lincei, (V), Vol. XIV, 1923.

2. F. Tricomi. Ulteriori ricerche sull'equazione $y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$. Rendiconti del Circolo matematico di Palermo, LII, 1928.

3. F. Tricomi. Ancora sull'equazione $y z''_{xx} + z''_{yy} = 0$. Rendiconti della Reale Accademia Nazionale dei Lincei, VI, 1927.

4. E. Holmgren. Sur un probleme aux limites pour l'equation $y^m\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$. Arkiv for Matematik, Astronomi och Fysik, 19 B. 1926.

5. E. Gursa. Course of mathematical analysis, Moscow-Leningrad, 1933.

6. Witteker-Watson. Course of contemporary analysis, Moscow-Leningrad, 1933.

7. S. Gellerstedt. Sur un problème aux limites pour une equation lineaire aux derivees partielles du second ordre de type mixte, Uppsala, 1935.

8. V. A. Fok. Airy's Tables, Moscow, 1946.

9. J. Kampe de Feriet. La fonction hypergeometrique, Paris, 1937.

F. Frankl. On the Theory of the Equation

$$y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

Summary [English-Language]

In this paper we solve two boundary problems for the equation which we call Darboux-Tricomi's equation after the authors who dealt with it.

$$y\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \qquad (1)$$

The solutions are sought on a domain lying within the semi-plane $y>0$, where the equation (1) is elliptique. The boundary of the domain is supposed to coincide with x-axis somewhere. As to the part of the boundary that lies within the semi-plane $y>0$, it is supposed to be sufficiently smooth and to approach the x-axis normally.

The following boundary problems are here considered:

1. Dirichlet's problem.

2. The problem in which the values of the unknown function are given on the part of boundary lying within the semi-plane $y>0$, while the normal derivatives are given on the part of the boundary which coincides with the x-axis.

These problems were already considered by F. Tricomi in [1], as well as in the papers [2] and [3], with the use of considerably complicated methods: the author either used Schwarz's alternating method or passed to the limit having proceeded from the domains lying, together with their boundaries, within the semi-plane $y>0$. In both cases Tricomi used two-dimensional integral equations of Fredholm's type.

The second problem was solved by S. Gellerstedt [7] by reducing it to one-dimensional Fredholm equations of the second kind. Gellerstedt, however, assumed that the ends of the curve L (see Fig. 1) coincided with arcs of a certain algebraic curve which Tricomi called "normal curve." In this paper we remove this restriction by means of some suitable estimates in both problems in question.

In Appendix we give three particular solutions of the equation (1) which refute some false assertions by Tricomi in [2].

DISTRIBUTION LIST

DEPARTMENT OF DEFENSE	NR. COPIES
HEADQUARTERS USAF	
ARL (ARB)	1
OTHER AGENCIES	
CIA	5
DIA	4
AID	2
NASA (ATSS-T)	1
OAR	1
OTS	2
NSA	6
ARMY (FSTC)	3
NAVY	3
NAFEC	1
AEC	2
RAND	1
AFCRL (CRXLR)	1

MAJOR AIR COMMANDS	NR. COPIES
DDC	20
AFSC	
SCFDD	1
TDBTL	5
TDBDP	2
TDGS	1
TDBMT	10
TDBXP	1
TDT	2
TDFCS (Robinson)	1
SSD (SSFI)	2
APGC (PGF)	1
ASD (ASFA)	2
ESD (ESY)	1
RADC (RAY)	1
AFWL (WLF)	1
AFETR (MTW)	1

BLANK PAGE